International Journal for Multidisciplinary Research (IJFMR)

E-ISSN: 2582-2160 ● Website: www.ijfmr.com ● Email: editor@ijfmr.com

Developments and Applications of Ramanujan Sums in Digital Signals and Image Compression: An Overview

Renjini S

Department of Mathematics T.K.Madhava Memorial College, Nangiarkulangara, University of Kerala, India.

Abstract

The Ramanujan sum (RS) has been used widely by mathematicians to derive infinite series expansions of many number theoretic functions.RS is a powerful technique having a wide range of applications in var- ious research domains. RS has many interesting properties,RS is integer valued , orthogonal ,periodic etc. Scientists make use of these properties of RS and has been applied this concept in different research fields like signal processing, analysis of biomedical signals and image compression .This paper gives an overview of the developments and applications of RS from the view point of digital signals , image transform and image compression

KEYWORDS: Ramanujan sum, Ramanujan Fourier Transforms, Ramanujan subspace,signal processing,image compression.

1 Introduction

Sreenivasa Ramanujan (22 December 1887-26 April 1920) was a great Indian mathematician .In his short life period he made immense contributions to mathe- matical analysis, number theory, infinite series and continued fractions. His genius was noticed by the legendary Cambridge mathematician , G.H Hardy. He recog- nised the significance of the formulae derived by Ramanujan in his letter and made arrangements for Ramanujan to collaborate with him at Cambridge University. His colloboration continued for 5 yeras until Ramanujan fell ill and return back to India.Ramanujan independently compiled nearly 3900 results in the form of identities and equations and he wrote many more formulae in colloboration with G.H.Hardy which fascinate mathematics even today[1, 2, 3, 23].

In 1918, Ramanujan[4] introduced a trignometric summation known as Ramanu- jan sum(RS). Due to its inherent orthogonal property, RS can be used to obtain convergent finite duration expressions for many number theoretic functions.Also owing to the integer property,periodicty and orthogonality RS can be used to simplify the computations of Arithmetic Fourier Transform(AFT)and Discrete Fourier Transform(DFT) coefficients for special type of signals and has been used in signal processing [7, 10, 11] and noise reduction[11, 17].RS has been applied to time frequency analysis[9], the acceleration of discrete Fourier transforms[8] and multi-resolution analysis[15].RS is used to calculate periodicity in presence of noise[11]and even a 2D period estimation can be achieved[12]..A class of Ramanu- jan operators based on Ramanujan sums have been

proposed in[14] and has applications in image processing noise level estimation.Energy efficient edge detection is introduced in[16] using approximate Ramanujan sums. RS has been developed further by introducing the concept of Ramanujan subspaces,Ramanujan dictionar- ies and Ramanujan filter banks which has application in identification of integer periodicities[19].Linear transforms using RS may be used for data compression[18] and work well for lossless compression of holographic data[31].
In this paper we present an overview of these developments of RS and some of its applications in yearwise. We exclude the proofs, it aims to give a lucid intro- duction so that the leaner may be inspired by this topic and read the references in depth and go deeper into the proofs and details.Section 2 describes the basics of Ramanujan Sum.In section 3, we review the concept of Ramanujan Fourier Transforms and its properties and Ramanujan subspaces in section 4. We discuss the milestones in the development of Ramanujan sums in digital signal processing and image compression and some of its applications in section 5.Finally we give conclusions in section 6.

2 Ramanujan Sum

Ramanujan sum (RS) ,is a trignometric summation introduced by Ramanujan in 1918[4] and has the form

$$c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} e^{\frac{i2\pi kn}{q}} = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} W_q^{-kn} \tag{1}$$

where (k, q)denotes the greatest common divisor(gcd) of k and q. Since the sum- mation runs over only those values of k that are coprime to q, the sum (1) has precisely $\phi(q)$ terms, and

$$c_q(n) = \phi(q) \tag{2}$$

From (1) we see that $c_q(n)$ is **periodic** with period q :

$$c_q(n+q) = c_q(n) \tag{3}$$

Equation(1) also shows that the **Discrete Fourier Transform (DFT)** of $c_q(n)$ is

$$C_q[k] = \begin{cases} q & \text{if } (k,q)=1 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Also the summation in (1) is *real valued* and this can be written as

$$c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} \cos\frac{2\pi kn}{q} \tag{5}$$

Furthermore it is *symmetric*,that is, $c_q(n) = c_q(-n)$. The first few Ramanujan sums are are given below

$$\begin{aligned} c_1(n) &= 1 \\ c_2(n) &= 1, -1 \\ c_3(n) &= 2, -1, -1 \\ c_4(n) &= 2, 0, -2, 0 \\ c_5(n) &= 4, -1, -1, -1, -1 \end{aligned}$$

Thus $c_q(n)$ is **inter-valued**.
Any two Ramanujan sums $c_{q1}(n)$ and $c_{q2}(n)$ are **orthogonal** in the sense that

$$\sum_{n=0}^{m-1} c_{q1}(n)c_{q2}(n) = 0, \quad q_1 \neq q_2 \tag{6}$$

where $m = lcm(q_1, q_2)$. Due to this orthogonality property, the DFT coefficients of two different amanujan sequences never overlap.

RS is defined as the sum of n^{th} powers of all q^{th} primitive roots of unity as W^{-k} is the primitive qth root of unity iff $(q, k) = 1$.

Here we mention some relations between Ramanujan sums and Mobius function $\mu(n)$ Ramnujan showed in[4] that

$$c_q(n) = \sum_{d/(q,n)} \mu(\frac{q}{d})q \tag{7}$$

In[23] Hardy proved that

$$c_q(n) = \frac{\mu(m)\phi(q)}{\phi(m)}, \quad m = \frac{q}{(q,n)} \tag{8}$$

Setting $n = 1$ we have $p = q$ so that

$$c_q(1) = \mu(q), \text{that is}, \quad \mu(q) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} W_q^k \tag{9}$$

. This yields that

$$c_q(n) = \mu(q), \quad \text{whenever}(q, n) = 1.$$

3 Ramanujan Fourier Transforms (RFT)

Several arithmetic functions in number theory can be expressed as linear combi- nations of $c_q(n)$, i.e.,

$$x(n) = \sum_{d=1}^{\infty} \alpha_q c_q(n) \tag{10}$$

The Ramanujan expansion in (10) has been derived for many number -theoretic arithmetic functions in [4]. Equation (10) is sometimes referred to as the Ra-manujan Fourier Transform expansion.Ramanujan Fourier Transform (RFT) is first introduced by Planat[7] in 2002. α_q are the RFT coefficients given by

$$\alpha_q = \frac{1}{\phi(q)}(\lim_{M\to\infty} \frac{1}{M} \sum_{n=1}^{M} x(n)c_q(n)) \tag{11}$$

4 Ramanujan Subspace

P.P.Vaidyanathan in[10] introduced the concept of Ramanujan subspace S_q using the Ramanujan sum $c_q(n)$.He defined the qXq integer circulant matrix as:

$$B_q = \begin{bmatrix} c_q(0) & c_q(q-1) & c_q(q-2) & \dots & c_q(1) \\ c_q(1) & c_q(0) & c_q(q-1) & \dots & c_q(2) \\ c_q(2) & c_q(1) & c_q(0) & \dots & c_q(3) \\ \vdots & & & & \\ c_q(q-2) & c_q(q-3) & c_q(q-4) & \dots & c_q(q-1) \\ c_q(q-1) & c_q(q-2) & c_q(q-3) & \dots & c_q(0) \end{bmatrix}$$

Every column is obtained by a circular downward shift of the previous column. The column space of B_q is called the *Ramanujan subspace* S_q.
B_q has rank $\phi(q)$.

5 Milestones in the development of Ramanujan sum in signal processing and image compression and its applica- tions

Ramanujan in[4] introduced the Ramanujan sum $c_q(n)$ and several standard arith- mertic functions in number theory has been expressed as linear combination of $c_q(n)$ as in equation(10) [1, 2].

5.1 Ramanujan sum in even symmetric signals

In 1950, Cohen[26, 27] investigated the even symmetric signals in number theory and observed that the DFT coefficients of this class of signals can be computed by forming integer-valued weighing coefficients of signals. It was later proved that these integer-valued coefficients are nothing but well-known Ramanujan sums (RS). RS can be expanded to get Ramanujan Fourier Transform (RFT).

5.2 Ramanujan sums for signal processing of low-frequency noise

The discrete Fourier transform (DFT)and fast Fourier transform (FFT) is well suited to the analysis of periodic or quasi periodic sequences, but fails to discover the constructive features of aperiodic sequences, such as low frequency noise. Planat[7] in 2002 introduced Ramanujan sums as a new signal processing tool for these experimental files. In contrast to the discrete Fourier transform in which the basis functions are all roots of unity , the Ramanujan-Fourier transform (RFT) is defined from powers over the primitive roots of unity . He provided a table of known RFT's and emphasized the newly discovered connection between 1/ f noise and arithmetic. Ramanujan–Fourier transform is proposed as a tool to analyse low-frequency noise. He used RFT as a method to analyse low-frequency noise in periodic, quasi-periodic, and complex time series as an alternative to DFT.

5.3 Ramanujan sums and discrete fourier transforms in special class of even symmetric functions

A special class of even-symmetric periodic signals was introduced in[8] in 2005. The signals are defined with respect to a fixed positive integer r for all integer values of the time index n. Let $\gcd(a, b)$ denote the greatest common divisor of the integers a and b . A signal $x_r(n)$ is called an even signal($mod r$) if

$$x_r(n) = x_r(gcd(n, r)), \text{for all n.} \tag{12}$$

The most distinctive feature of these signals is that their real-valued Fourier coef- ficients can be calculated by forming a weighted average of the signal values using integer-valued coefficients. The signals arise from number-theoretic concepts con- cerning a class of functions called even arithmetical functions. The integer-valued weighting coefficients, being sums of complex roots of unity, are the Ramanujan sums and may be computed recursively or through closed-form arithmetical rela- tions. Let us denote by $\tau(r)$the total number of positive divisors of r, including 1 and r . For each positive divisor of r, we can construct what is called a reduced residue system. The set

$$S_d = \{\frac{r}{d}U | gcd(U,d) = 1, 0 \leq U < d\}$$

contains representative members of such a reduced residue system. Based on the concept of reduced residue systems, the class of even signals ($modr$) may be additively expressed as the linear combination of $\tau(r)$ simple (0, 1)-signals $h_{r,d}(n)$. Specifically, we can write

$$x_r(n) = \sum_{\substack{d|r \\ d\geq 1}} x_r(\frac{r}{d}) h_{r,d}(n)$$

where the (0,1)-signals $h_{r,d}(n)$ is periodic with period r, which is defined for $n \in [0, r-1]$ as

$$h_{r,d}(n) = \begin{cases} 1, & \text{if } n \in S_d \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that the signals $h_{r,d}(n)$ are even ($modr$). The DFT of $h_{r,d}(n)$ is computed using a sum of the form

$$H_{r,d}(n) = \sum_{k=0}^{r-1} h_{r,d}(k) W_r^{-nk}$$

where $W_r = exp(\frac{j2\pi}{r}$ and $j = \sqrt{(-1)}$. This can be rewrire as

$$H_{r,d}(n) = \sum_{\substack{0\leq k\leq r-1 \\ k=\frac{r}{d}U \\ gcd(U,d)=1}} W_r^{-nk} = \sum_{\substack{0\leq U\leq d-1 \\ gcd(U,d)=1}} W_d^{-nU}$$

The last complex sum in this equation is the *Ramanujan sum* $c_d(n)$ and

$$H_{r,d}(n) = c_d(n)$$

The recursive method of computation is based on the cyclotomic polynomials . If the signal values are integers, the computation of the discrete Fourier trans- form (DFT) coefficients of this class of signals can be performed in an exact quantization-error-free manner by performing arithmetical operations on integers.

5.4 Ramanujan sum in a special class of odd-symmetric length-4N periodic signals

In 2007 Pei S-C.et al.defined a class of odd-symmetric periodic signals and showed how the odd Ramanujan sums are used as weighting coefficients to compute their pure imaginary Discrete Fourier Transform integer-valed coefficients. A signal $x_r(n)$ is called an odd signal ($modr$) if

$$x_r(n) = \begin{cases} x_r(gcd(n,r)), & \text{if} \frac{r}{gcd(n,r)} \equiv 0 mod 4, \text{and} \frac{n}{gcd(n,r)} \equiv 1 mod 4 \\ -x_r(gcd(n,r)), & \text{if} \frac{r}{gcd(n,r)} \equiv 0 mod 4, \text{and} \frac{n}{gcd(n,r)} \equiv 3 mod 4 \\ 0, & \text{elsewhere.} \quad \forall n \end{cases}$$

. Also we write

$$x_r(n) = \sum_{\substack{d|r \\ d\geq 1}} x_r(\frac{r}{d}) h_{r,d}(n)$$

where the signal $h_{r,d}(n)$ is periodic with period r, which is defined for $n \in [0, r-1]$ as

$$h_{r,d}(n) = \begin{cases} 1, & \text{if } \frac{r}{gcd(n,r)} = d \equiv 0 \, mod 4, \frac{dn}{r} \equiv 1 mod 4 \\ -1, & \text{if} \frac{r}{gcd(n,r)} = d \equiv 0 \, mod 4, \frac{dn}{r} \equiv 3 \, mod 4 \\ 0, & \text{elsewhere} \end{cases}$$

. The DFT of $h_{r,d}(n)$ is computed using a sum of the form

$$H_{r,d}(n) = \sum_{0\leq k\leq r-1} h_{r,d}(k) W_r^{-nk}$$

and is computed as

$$H_{r,d}(n) = i. \sum_{\substack{0\leq k'\leq d-1 \\ gcd(d,k')=1 \\ d\equiv 0 mod 4}} W_d^{-k'(n+\frac{d}{4})}$$

, where $k' = \frac{dk}{r}$ .The last complex sum is another odd type of the Ramanujan sums and we write it as

$$H_{r,d}(n) = i.c_d(n + \frac{d}{4}), d \geq 1, \qquad d \equiv 0 \, mod 4$$

In[6] they combined the odd part with the even part. Circular shift was also introduced, and proved that this method is better than that, theoretically and practically. They have shown that a special class of odd periodic signals possess a DFT representation, with the imaginary integer-valued odd Ramanujan sums playing the role of the complex-valued roots of unity. In addition, combining this odd part with the even part as mensioned above, one can get more dimensions.The odd Ramanujan sum, being the sums of complex roots of unity, can be calculated either using closed-form formulas or computed recursively through the impulse response of a derived infinite impulse response (IIR) filter. This special class of odd-symmetric signals and odd Ramanujan sums can be combined together with the previous even-symmetric special class signals and the well-known even Ramanujan sums as a useful tool for signal processing.

5.5 Analysis of long period sequences and noise by Ramanujan sums

The unavailability of nonstatistical model of 1/f noise due to its randomness has motivated M. Planat et al. to work on it. In their work [13], it is shown that using RS, arithmetical functions can be analyzed better as RFT has the ability to extract quasi periodic features and fine aperiodic features (low-frequency signal) that the DFT fails to show . As RS analysis uses the properties of irreducible functions, it is favorable to analyze rich time series signal showing a $\frac{1}{f^{\alpha}}$ $(0 < \alpha < 2)$ FFT dependence. The authors established their claim by showing two examples: first on the data from the stock market (for which the price index FFT follows a $\frac{1}{f^2}$ -law) and in the second one on the data obtained from solar cycle activity (for which the coronal index follows a $\frac{1}{f}$). Thus, RFT proved to be a magnifying glass for analyzing 1/f noise.

5.6 Doppler spectrum estimation by Ramanujan Fourier Transform

The usability of Ramanujan Fourier transform has been expanded by Lagha and Bensebti in 2009 in estimation of the Doppler spectrum for weather radar signal.In[29],they studied the Ramanujan sums, $c_q(n)$,in order to estimate the spectrum properties of pulsed Doppler radar signals, used in meteorology.The Doppler spectrum estimation of a weather radar signal in a classic way can be made by two methods, temporal one based in the autocorrelation of the successful signals, whereas the other one uses the estimation of the power spectral density PSD by using Fourier transforms. Lagha and Bensebti introduced a new tool of signal processing based on Ramanujan sums $c_q(n)$, adapted to the analysis of arithmetical sequences with several resonances $\frac{p}{q}$. These sums are almost periodic according to time n of resonances and aperiodic according to the order q of resonances. The RamanujanTransform facilitates the distinction of low-level weather modes like low intensity winds or rains.

5.7 Time frequency analysis via Ramanujan sums

In 2012, Sugavaneswaran et al. introduced an extension to time–frequency analy- sis of signals from 1-D RFT into 2-D space[9]. A novel class of Ramanujan Fourier Transform (RFT) based time-frequency(TF) transform functions, constituted by Ramanujan sums (RS) basis was introduced. A modified ambiguity function (AF) representation is computed by using the RS (on the time-varying autocorrelation values) as

$$\hat{A}(q,\tau) = \int x(t-\frac{\tau}{2})x^*(t+\frac{\tau}{2})c_q(t)dt. \quad (13)$$

. Such a computation results in an overall reduction in the number of ambiguity domain (AD) coefficients. These AD-mapped coefficients can then be transformed into a sparser time-frequency representation, using the RFT computed from their characteristic RS.The proposed special class of transforms offer high immunity to noise interference, since the computation is carried out only on co-resonant components, during analysis of signals. Further a 2-D formulation of the RFT for analyzing the AD coefficients is defined as

$$x_q = \frac{1}{\phi(q)} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x(n)c_q(n) \tag{14}$$

$$x_{q_1,q_2} = \frac{1}{\phi(q_2)} \lim_{N_2\to\infty} \frac{1}{N_2} \sum_{\tau=1}^{N_2} R_1 c_{q_2}(n) \tag{15}$$

where

$$R_1 = \frac{1}{\phi(q_1)} \lim_{N_1\to\infty} \frac{1}{N_1} \sum_{\theta=1}^{N_1} A_c(\theta,\tau)c_{q_1}(n)$$

Experimental validation using synthetic examples, indicates that this technique shows potential for obtaining relatively sparse TF-equivalent representation and can be optimized for characterization of certain real-life signals. The 2-D inter- pretation is then used to project the data from ambiguity domain (AD) into the time-frequency space, in place of the standard DFT. Following this, the dimen- sionality reduction capabilities of the deduced transform for certain signal classes can be calculated.

5.8 Ramanujan Sums-Wavelet Transform

The wavelet transform decomposes a signal into a multiresolution representa- tion. It is localized in both the time and the frequency domains. The discrete wavelet transform (DWT) is any wavelet transform for which the wavelets are discretely sampled. In 2013, Chen et al. proposed a new transform by combining the wavelet transform with the RS transform called *Ramanujan Sums-Wavelet Transform(RSWT)*.In [5], they have proposed to combine the wavelet transform with the RS transform in order to provide a new way to represent signals and is defined as

$$RSW(n,a,q) = \frac{1}{\sqrt{|a|}} \sum_{t=1}^{T} f(t)c_q(t)\psi(\frac{t-n}{a}) \tag{16}$$

$$= \frac{1}{\sqrt{|a|}} \sum_{t=1}^{T} g(t,q)\psi(\frac{t-n}{a}) \tag{17}$$

he new transform takes advantage of both the multiresolution property of the wavelet transform and the orthogonality property of the RS transform.They anal- ysed the properties of this transform by examples for a few very simple cases and this transform can provide a new representation for signal analysis and other related applications such as as time-frequency analysis,pattern recognition and image analysis.

5.9 Ramanujan sums in signal processing

In 2014, Vaidyanathan replaced each $c_q(n)$ with an entire subspace of period- q signals which he call the Ramanujan subspace that we discussed in section

4. This space has dimension $\phi(q)$(Euler's totient function)and contains $c_q(n)$ as one member. In [10] he studied several properties of this subspace, such as periodicity properties, autocorrelation properties, and projection properties. The results obtained in this paper lead to the developments reported in the companion paper [11] which introduced two new representations for finite duration (FIR) signals in terms of Ramanujan sums.The first one is

$$x(n) = \sum_{q=1}^{N} a_q c_q(n), \quad 0 \le n \le N-1 \tag{18}$$

, where $c_q(n)$ is the Ramanujan sum. The second one, called the Ramanujan Periodic Transform(RPT

$$x(n) = \sum_{q_i/N} \sum_{l=0}^{\phi(q_i)-1} \beta_{il} c_{q_i}(n-l), \tag{19}$$

where q_i are divisors of N and c_{qi} (n) is the q_i th Ramanujan sum is based on Ramanujan spaces. These are somewhat similar to time-frequency representa- tions. The most important point is that an arbitrary finite duration signal can be decomposed into a sum of a finite number of orthogonal projections $x_{qi}(n)$ be- longing to Ramanujan subspaces S_q , with each component representing aperiodic component of $x(n)$. This is useful in extracting hidden periods in finite duration signals. He also demonstrated a denoising application, which was based upon the result that the circular correlation of a signal has the same RPT structure as the original signal.

5.10 Ramanujan filter banks for estimation and tracking of periodic- ities

A discrete time signal $x(n)$ is said to be periodic with period P if P is the smallest positive integer such that

$$x(n+P) = x(n) \quad \forall \quad n \in \mathbb{Z}$$

Vaidyanathan and Tenneti in 2015 proposed a new filter-bank structure for the estimation and tracking of periodicities in time series data. These filter-banks are inspired from recent techniques on period estimation using high-dimensional dic- tionary representations for periodic signals. Apart from inheriting the numerous advantages of the dictionary-based techniques over conventional period-estimation methods such as those using the DFT, the filter-banks proposed in[19] expand the domain of problems that can be addressed to a much richer set. For instance, we can now characterize the behavior of signals whose periodic nature changes with time. This includes signals that are periodic only for a short duration and signals such as chirps. For such signals, they used a time vs period plane analogous to the traditional time vs frequency plane. They showed that such filter banks have a fundamental connection to Ramanujan Sums and the Ramanujan Periodicity Transform. The property that makes the Ramanujan filters useful is their support in the frequency domain. We cannot replace $c_q(n)$ with an arbitrary q periodic impulse response and expect the lcm property to still hold. Nevertheless, we can construct a family of filters with the same support by convolving the Ramanujan sums $c_q(n)$ with arbitrary sequences whose spectrum is not zero at the frequencies , where $gcd(k_i, q) = 1$. Such filter banks might be of interest in various real- world settings, for instance in detecting seizures using EEG data or to identify repetitive structures in protein molecules .The paper[24] also shows how Ramanu- jan filter banks can be used to generate time-period plane plots which track the presence of time varying, localized, periodic components.

5.11 Two-dimensional period estimation by Ramanujan sum

Period estimation in one dimension (1-D) has been studied for years. However, 2-D period estimation is a hard problem since it has three parameters to deter- mine: length, width, and direction. In 2016, a special kind of 2-D function called 2D-gcd-delta function is proposed in[30]. A 2-D-gcd-delta function denoted by

$S_{N,d_1,d_2,t}(n_1, n_2)$ is defined as

$$S_{N,d_1,d_2,t}(n_1,n_2) = \begin{cases} 1, & \text{if} \quad gcd(n_1,N) = d_1, gcd(n_2,N) = d_2, \frac{n_1}{d_1} - t\frac{n_2}{d_2} \equiv 0 \,(mod\, d) \\ 0 & elsewhere \end{cases} \tag{20}$$

where $d = gcd(N_1, N_2)$, $N_1 = \frac{N}{d_1} and\, N_2 = \frac{N}{d_2}$, $n_1, n_2 = 0 \sim N - 1 and\, t \in I_d$.

It has close relationship with Ramanujan's sum and the 2-D periodicity matrix.

In 2017, Pei and Chang described how to use this function to decompose an im- age into subband signals [12]. Each subband signal will have its own periodicity matrix so they also provided a simple algorithm to calculate the least common multiple (LCM) of those subband signal periodicity matrices. Concrete experi- ments are given by the authors to prove the robustness of the proposed 2-D period estimation method.

5.12 Ramanujan class of operators for edge detection and estimation of noise level

In 2018,D.K.Yadav et al,defined a class of operators, based on the Ramanujan Sums, termed as *Ramanujan class of operators* and proved that these operators

possess properties of first derivative and with a particular shift, of second deriva- tive also in [14]. For a given q, $(q \neq 0)$ Ramanujan operator denoted by $\hat{R}_q$ and is defined as a linear operator whose kernal is given as one period of $c_q(n)$, which is denoted by $\hat{c}_q(n)$ and

$$\hat{c}_q(n) = \begin{cases} c_q(n) & 0 \leq n \leq q-1 \\ 0 & \text{otherwise} \end{cases}$$

Let $x(n)$ be any arbitrary signal.Application of Ramanujan operator on this signal would result in

$$(\hat{R}_q x)(n) \triangleq \sum_k \hat{c}_q(k) x(n-k)$$

In image processing Ramanujan operators can be used to find edges in an image. Applying Ramanujan operator on an image f in horizontal direction we can get edges in vertical direction and similarly applying same operator in vertical direc- tion we can get edges in horizontal direction. Another application of Ramanujan operator is to estimate the noise level in a given signal. It is well known that the derivatives are sensitive to noise. Since Ramanujan operators behave as first derivative, they can be used in estimation of noise level.

5.13 Energy-Efficient Edge Detection using Approximate Ramanujan Sums

The Ramanujan operator defined in section5.12 can be used for edge detection for its differential property.In 2020, for the first time an approximate comput- ing based energy-efficient hardware accelerator using Ramanujan Sums for edge

detection applications was introduced . In [16],they exploit the inherent error resilience in the edge detection algorithm to propose an approximation technique by combining two very efficient approximation methods, viz., precision scaling and loop skipping that reduces the energy consumption of the edge detection system. A gradient descent based novel heuristic was designed to automatically configure the two approximation knobs to result in the least energy consumption for a specified application-level quality, that reduces the energy consumption of the total accelerator by almost 30% for negligible applicationlevel quality degra- dation. The energy savings increase to a range of as much as 70% − 80% for
5 − 20% quality degradation.This is the first work to show hardware realization
of an Approximate Ramanujan Sums operator based edge detection algorithm.

5.14 Lossless compression of holograms using Ramanujan Sums

In [18],G. Chen introduced matrix based Ramanujan sums transforms for 1-D and 2-d signals and its inverses. They defined the matrix

$$A(q,j) = \frac{1}{\phi(q)M} c_q(mod(j-1,q)+1) \tag{21}$$

where $q, j \in [1, M]$ *and* mod ()means the modular operation. The input signal can be represented as $X = (x(1), x(2), \cdots, x(M))^T$, where the T means the transpose of the vector.

The forward 1-D RS transform of a signal X can be defined as

$$Y = AX$$

, where $Y = (y(1), y(2), \cdots, y(M))^T$. the inverse 1-D RS transform can be obtained as

$$X = A^{-1}Y,$$

where A^{-1} is the inverse if matrix A.

For 2-D images, the forward RS transform is

$$Y(p,q) = \frac{1}{\phi(p)\phi(q)} \frac{1}{MN} \sum_{m=1}^{N} \sum_{n=1}^{M} x(m,n) c_p(n) c_q(n) \tag{22}$$

This can be written in the matrix form

$$Y = AXA^T$$

,where $X = (x(m,n)) form \in [1, M] and n \in [1, N]$. The inverse 2-D RS transform can be given as

$$X = A^{-1}Y(A^{-1})^T$$

In [31] H.Shiomi et.al introduced a lossless hologram-compression method that employs transforms using the Ramanujan sums based on the matrix based Ra- manujan sums transforms(RST) mentioned above. In this article, the authors compared the compression ratios of four kinds of hologram data sets such as

Complex hologram generated from point-cloud data

Amplitude hologram generated from point-cloud data

Phase hologram generated from point-cloud data

Light waves propagated from a natural image with random phases both with and without using

Ramanujan-sums-based transforms like Ramanujan FIR Trans- form(RFT),Ramanujan Periodic Transform (RPT),Ramanujan SumsTransform (RST) etc. The compression ratio is defined by

$$Compression\ ratio = \frac{Compressed\ data\ size}{Original\ data\ size} X100\%$$

.They found that the Ramanujan periodic transform improves the compression ratio of hologram data when using data having prime number dimensions.

6 Conclusion

Ramanujan sums (RS), which originated as a number-theoretic tool for express- ing arithmetical functions, have evolved into a powerful mathematical framework with wide-ranging applications in modern engineering disciplines. Beyond their theoretical elegance, RS has proven to be instrumental in areas such as signal processing, information theory, spectrum estimation, coding theory, and even biomedical fields. The Ramanujan Fourier Transform (RFT) and Ramanujan Periodic Transform (RPT) offer distinct advantages over conventional methods like the Discrete Fourier Transform (DFT), particularly in their ability to uncover hidden periodicities in signals, where traditional approaches often fall short. One of the key strengths of RFT and RPT is their integer-based operations, which simplify computation, reduce the complexity associated with real and complex coefficients, and minimize quantization errors. This integer framework makes RS-based algorithms not only computationally efficient but also highly suitable for hardware implementation, offering reduced word lengths, faster processing times, and lower costs. Additionally, the reconfigurability of RS-based hardware systems enhances their versatility, allowing for adaptation across a variety of applications. The potential for RS to streamline both software and hardware solutions while improving accuracy and performance is significant, and ongoing research is ac- tively exploring its application in new and emerging fields. As RS continues to bridge the gap between abstract mathematics and practical engineering, its role in shaping future innovations is poised to expand even further.

References

1. Hardy G. H, Seshu Iyer P. V. and Wilson B. M., *Collected papers of Srinivasa Ramanujan*, Cambridge University Press, London, 1927.
2. Berndt B. C.,*Ramanujan's note books*, Springer-Verlang, Inc., N. Y., 1991.
3. Andrews G. E.,Berndt B. C., *Ramanujan's lost notebook*, Springer, N. Y., 2005.
4. Ramanujan S.,*On some trignometrical sums and their applications in the the- ory of numbers*, Trans. Cambridge Philosoph. Soc., volXxIi, no. 13,pp 259-276, 1918.
5. Guangyi Chen, Krishnan S, Wenfang Xie, *Ramanujan sums wavelet transform for signal analysis*, Wavelet Analysis and Pattern Recognition(ICWAPR)2013
6. Soo-Chang Pei; Kuo-Wei Chang,*Odd Ramanujan Sums of complex roots of unity* , IEEE Signal Process. Lett., vol. 14,pp. 20-23,Jan. 2007.
7. M. Planat, *Ramanujan sums for signal processing of low frequency noise* , in Proc. IEEE Int. Freq. Contr. Symp. PDA exhib.,pp. 715-720, 2002.
8. S. Samadi. M, Omar Ahamad, and M. N. S. Swamy, *Ramanujan Sums and Discrete Fourier Transforms*, IEEE Signal Process. Lett., vol. 12, pp. 293- 296,April 2005.

9. L. Sugavaneswaran, S. Xie, K. Umapathy and S. Krishnan, *Time frequency analysis via Ramanujun sums*, IEEE Signal Process. Letter, vol.19,pp. 352- 355,June 2012.
10. Vaidyanathan P. P., *Ramanujan sums in the context of signal processing Part I: fundamentals*, IEEE Trans. sign. process, vol. 62, pp.4145-4157,2014.
11. Vaidyanathan P. P., *Ramanujan sums in the context of signal processing Part II: FIR representations and applications.*, IEEE Trans. sign. process, vol. 62, pp.4158-4172,2014.
12. Soo-Chang Pei; Kuo-Wei Chang,*Two Dimensional Period Estimation by Ra- manujan's sum*, IEEE Sig. Proc.,Volume: 65 Issue: 19,2017.
13. M. Planat, M. Minarovjech and M. Saniga, *Ramanujan sums analysis of long period sequences and noise*, EPLJ. (Europhysics Letters), vol. 85,no. 4,pp.
14. 40005:1-5,2009.
15. Devendra Kumar Yadav, Gajraj Kuldeep and SD Joshi, *Ramanujan sums and derivatives and applications*, IEEE Signal Processing Letters, 25(3):413- 416,2018.
16. Devendra Kumar Yadav, Gajaraj Kuldeep, S. D. Joshi, *Orthogonal Ramanu- jan Sums, its properties and Applications in Multiresolution Analysis*, IEEE Trans. Signal. Process., vol. 66, no. 21,pp 5789-5798,Nov. 2018.
17. A. Ghosh,K. Gaurav Kumar, D. De, Arnab Raha and Mrinal Kanti Naskar, *Energy-Efficient Edge Detection using Approximate Ramanujan Sums*, 21st International Symposium on Quality Electronic Design (ISQED),Santa Clara, CA, USA, 25-26 March 2020.
18. A. M. Solomon, D. Abraham, A. George and M. Manuel, *Signal denoising using Ramanujan periodic transform*, in Proc. 2nd Int. Conf. Trends Electron. Informat(ICOEI),pp. 51-56, May 2018.
19. G. Chen, S. Krishnan and T. D. Buli, *Matrix- based Ramanujan -sums trans- forms*, IEEE Signal Process, Lett. Vol. 20,no. 10,pp 942-944,Oct. 2013.
20. Tenneti S,Vaidyanathan P P,*Ramanujan filter banks for estimation and trac- ing of periodicities*,IEEE Intl.Conf.Acoust.,Speech,and Sig.Proc., 19-24 April
21. ,2015,Austalia.
22. P. P. Vaidyanathan and S. Tenneti, *Sreenivasa Ramanujan and signal pro- cessing problems*, Phil. Trans. Royal Soc., 2019.
23. Debaprasad De, K. Gaurav Kumar, A. Ghosh, M. K. Naskar, *Ramanujan Sums and Signal Processing: An Overview*, Proceedings of the second Inf. Conf. on Microelectronics, Computing and Communication Systems, 2017.
24. Hardy G. H. *Notes on Ramanujan's trignometric function*$c_q(n)$ *and certain series of arithmetical functions*, Proc. Camb. Phil. Soc. 20,263-271,1921.
25. Hardy G. H., Write E. M., *An introduction to theory of numbers.*, Oxford University Press, NewYork, 2008.
26. Vaidyanath P P,Tenneti s, *Properties of Ramanujan filter banks*,European Sig.Proc.Conf.,Nice,France,31 August -4 September,2015.
27. S.-Chang Pei, C.-Chang Wen, *Legendre Ramanujan sums transform* 23rd Eu- ropean Sign. Proc. Conf. EUSIPCO), 2015.
28. E. Cohen, *A class of arithmetical functions*, Proc. Nat. Acad. Sci. U.S.A., vol. 41,pp. 939–944,1955.
29. E. Cohen, *Representations of even functions (modr). I. Arithmetical identi- ties*,Duke Math.

J., vol. 25, pp. 401–421, 1958.
30. H G Gadiyar and R Padma, *Ramanujan -Fourier series,the Wiener- Khintchine formula and the distribution of prime pairs*,Physica A 269,503- 510;2006.
31. Lagha M., Bensebti M., *Doppler spectrum estimation by Ramanujan Fourier transform (RFT)*. Digital Signal Processing 19, pp. 843–851,2009.
32. S.C.Pei and K.W.Chang,*Integer 2-D Discrete Fourier Transform pairs and eigen vectors using Ramanujan's sum*, IEEE Signal Processing Let- ters,vol.23,no.1,pp 70-74,Jan 2016.
33. Harutaka Shiomi , Tomoyoshi Shimobaba, Takashi Kakue , Tomoyosh Ito,*Lossless Compression Using the Ramanujan Sums: Application to Holo- gram Compression*,IEEE Access (Volume: 8),Page(s): 144453 - 144457,2020.