International Journal for Multidisciplinary Research (IJFMR)

E-ISSN: 2582-2160 • Website: www.ijfmr.com • Email: editor@ijfmr.com

On Hyperbolic Hsu-Structure Manifold, Recurrent and Symmetry

Dr S.S.Dhapola

Head In charge, Assistant Professor Dept of Mathematics, Govt.P.G.College Bageshwer

Abstract- In this paper, we have defined recurrence and symmetry of different kinds in H- Hsu-structure manifold. Some theorem establishing relationship between different kinds of recurrent H- HSU-Structure manifold involving equivalent conditions with respect to projective, conformal, conharmonic and concircular curvature tensors has been discussed recurrence parameter have also been studied. Index Terms- Recurrence parameter,Curvature Tensors,C^{∞} -function,Hsu-structure manifold.

1.Introduction

If a differentiable manifold Vn, of differentiability class C^{∞}.there be in Vn,a vector valued linear function F of classC^{∞}, satisfying the algebraic equation

$\bar{\bar{X}}$ = - $a^r X$, for arbitrary vector field X . (1.1)

Where $\bar{X}$= FX , 0≤ $r \leq n$ and 'a'is real or imaginary number,then{F} is said to give to Vn a Hyperbolic HSU-structure defined by the equations(1.1) and the manifold Vn is called a Hyperbolic HSU –structure manifold.Hyperbolic HSU-structure manifold or briefly H-Hsu-structure manifold.

Remark(1.1) : The equation (1.1) gives different structures for different values of 'a' and r .

If a= ± 1 and r =2 ,it is an almost complex structure.

If a= $\pm i$ and r = 2 , it is an almost product structure or a hyperbolic almost complex structure.

If a = 0, it an almost tangent structure or almost hyperbolic tangent structure.

If a≠0, s It is the hyperbolic π-structure.

Let the Hsu – structure Vn, be endowed with a Hermitian metric tensor g, such that

g($\bar{X}$,$\bar{Y}$) –a^r g(X,Y) =0,

Then{F,g} is said to give Vn a hyperbolic Hsu-structure metric manifold.

Agreement (1.1):In what follows and the above, the equations containing X,Y,Z…………,etc. hold for these arbitrary vector in Vn,

The curvature tensor K,a vector –valued tri-linear function w.r.t the connexion D is given by

$$K(X,Y,Z) = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z, \quad (1.2)a$$

Where

$[X,Y] = D_X Y - D_Y X$ (1.2)b

The Ricci tensor in V_n is given by

$\mathrm{Ric}(Y,Z) = (C_1^1 K)(Y,Z).$ (1.3)

Where by $(C_1^1 K)(Y,Z)$, we mean the contraction of $K(X,Y,Z)$ with respect the first slot.
For Ricci tensor, we also have
$Ric(Y,Z) = Ric(Z,Y),$ (1.4)a

$Ric(Y,Z) = g(r(Y),Z) = g\big(Y,r(Z)\big),$ (1.4)b

$(C_1^1 r) = R$ (1.4)c

Let W,C,L and V be the Projective,Conformal, conharmonic and concircular curvature tensors respectively given by

$W(X,Y,Z) = K(X,Y,Z) - \frac{1}{(n-1)}[Ric(Y,Z)X - Ric(X,Z)Y]$ (1.5)

$C(X,Y,Z) = -\frac{1}{(n-2)}\{Ric(Y,Z)X - Ric(X,Z)Y - g(X,Z)r(Y) + g(Y,Z)r(X)\} + \frac{R}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y].$ (1.6)

$L(X,Y,Z) = K(X,Y,Z) - \frac{1}{(n-2)}[Ric(Y,Z)X - Ric(X,Z)Y - g(X,Z)r(Y) + g(Y,Z)r(X)].$ (1.7)

$V(X,Y,Z) = K(X,Y,Z) - \frac{R}{n(n-1)}[g(Y,Z)X - g(X,Z)Y].$ (1.8)
A manifold is said to be recurrent, if

$(\nabla K)(X,Y,Z,T) = A(T_1)K(X,Y,Z).$

The recurrent manifold is said to be symmetric, if
$A(T_{1)} = 0$, in the equation (1.9).

II RECURRENCE AND SYMMETERY OF DIFFERENT KINDS

Let Q, a vector – valued trilinear function be any one of the curvature tensors K,W,C,L or V. Then we will define recurrence of different kinds as follows:
Definition(2.1). A –HSU-structure manifold is said to be (1)-recurrent in Q,if
$a^r(\nabla Q)(X,Y,Z,T) - Q((\nabla F)(\bar{X},T),Y,Z) = a^r P_1(T)Q(X,Y,Z)$ (2.1)

Where P_1(T) is non – vanishing C^∞ function.

Definition(2.2). A H-HSU- structure manifold is said to be (2)- recurrent inQ,if

$$a^r(\nabla Q)(X,Y,Z,T) - Q(X,(\nabla F)(\bar{Y},T),Z) = a^r P_1(T)Q(X,Y,Z) , \quad (2.2)$$

Definition(2.3). A H-HSU- structure manifold is said to be (3)- recurrent inQ,if

$$a^r(\nabla Q)(X,Y,Z,T) - Q(X,Y(\nabla F)(\bar{Z},T) = a^r P_1(T)Q(X,Y,Z) , \quad (2.3)$$

Definition(2.4). A H-HSU- structure manifold is said to be Ricci- (1)- recurrent ,if

$$a^r(\nabla \mathrm{Ric})(,Y,Z,T) - Ric((\nabla F)(\bar{Y},T),Z) = a^r P_1(T)Ric(Y,Z) , \quad (2.4)$$

Definition(2.5). A H-HSU- structure manifold is said to be Ricci- (2)- recurrent ,if

$$a^r(\nabla \mathrm{Ric})(,Y,Z,T) - Ric(Y,(\nabla F)(\bar{Z},T) = a^r P_1(T)Ric(Y,Z) , \quad (2.5)$$

Definition(2.6). A –HSU-structure manifold is said to be (1,2)-recurrent in Q,if

$$a^r(\nabla Q)(X,\bar{Y},Z,T) - Q\big((\nabla F)(\bar{X},T),\bar{Y},Z\big) + a^r Q(X,(\nabla F)(Y,T),Z)$$
$$= a^r P_1(T)Q(X,\bar{Y},Z), \quad (2.6)a$$

or equivalently

$$a^r(\nabla Q)(\bar{X},Y,Z,T) - Q(\bar{X}(\nabla F)(\bar{Y},T),Z) + a^r Q\big((\nabla F)(X,T),Y,Z\big)$$
$$= a^r P_1(T)Q(\bar{X},Y,Z), \quad (2.6)b$$

or equivalently

$$a^{2r}(\nabla Q)(X,Y,Z,T) - a^r(Q\big((\nabla F)(\bar{X},T),Y,Z\big) - a^r Q(X,(\nabla F)(\bar{Y},T),Z)$$
$$= a^{2r} P_1(T)Q(X,Y,Z), \quad (2.6)c$$

Definition (2.7). A –HSU-structure manifold is said to be (1,2)-recurrent in Q,if

$$a^r(\nabla Q)(X,Y,\bar{Z},T) - Q\big((\nabla F)(\bar{X},T),Y,\bar{Z}\big) + a^r Q\big(X,Y,(\nabla F)(Z,T)\big)$$
$$= a^r P_1(T)Q(X,Y,\bar{Z}), \quad (2.7)a$$

or equivalently

$$a^r(\nabla Q)(\bar{X},Y,Z,T) - Q(\bar{X},Y(\nabla F)(\bar{Z},T\ )) + a^r Q\big((\nabla F)(X,T),Y,Z\big)$$
$$= a^r P_1(T)Q(\bar{X},Y,Z), \quad (2.7)b$$

or equivalently

$$a^{2r}(\nabla Q)(X,Y,Z,T) - a^r(Q\big((\nabla F)(\bar{X},T),Y,Z\big) - a^r Q\big(X,Y(\nabla F)(\bar{Z},T)\big)$$
$$= a^{2r} P_1(T)Q(X,Y,Z), \quad (2.7)c$$

Definition(2.8). A –HSU-structure manifold is said to be (2,3)-recurrent in Q,if

$$a^r(\nabla Q)(X,Y,\bar{Z},T) - Q(X,(\nabla F)(\bar{Y},T),\bar{Z}) + a^r Q\big(X,Y,(\nabla F)(Z,T)\big)$$
$$= a^r P_1(T)Q(X,Y,\bar{Z}), \quad (2.8)a$$

or equivalently

$a^r(\nabla Q)(X,\bar{Y},Z,T) - Q(X,\bar{Y},(\nabla F)(\bar{Z},T\ ) + a^r Q(X,(\nabla F)(Y,T),Z)$
$= \ a^r P_1(T)Q(X,\bar{Y},Z),$ (2.8)b

or equivalently

$a^{2r}(\nabla Q)(X,Y,Z,T) - a^r(Q(X,(\nabla F)(\bar{Y},T),Z) - a^r Q(X,(\nabla F)(Y,T),\bar{Z})$
$= \ a^{2r} P_1(T)Q(X,Y,Z).$ (2.8)c

Definition(2.9). A H-HSU- structure manifold is said to be Ricci- (1,2)- recurrent ,if

$a^r(\nabla \mathrm{Ric})(,Y,\bar{Z},T) - Ric\big((\nabla F)(\bar{Y},T),Z\big) + a^r Ric(Y,(\nabla F),Z,T))$
$= a^r P_1(T)Ric(Y,\bar{Z})$, (2.9)a

or equivalently

$a^r(\nabla \mathrm{Ric})(,\bar{Y},Z,T) - Ric\big(\bar{Y},(\nabla F)(\bar{Z},T)\big) + a^r Ric((\nabla F)(Y,T),Z)$
$= a^r P_1(T)Ric(\bar{Y},Z),$ (2.9)b

Or equivalently

$a^{2r}(\nabla \mathrm{Ric})(Y,Z,T) - a^r(Ric\big((\nabla F)(\bar{Y},T),Z\big) - a^r Ric(\bar{Y},(\nabla F)(Z,T)$
$= \ a^{2r} P_1(T)Ric(Y,Z).$ (2.9)c

Definition(2.10). A –HSU-structure manifold is said to be (1,2,3)-recurrent if

$a^r(\nabla Q)(X,\bar{Y},\bar{Z},T) - Q\big((\nabla F)(\bar{X},T),\bar{Y},\bar{Z}\big) + a^r Q\big(X,\bar{Y},(\nabla F)(Z,T)\big)$
$+a^r Q(X,(\nabla F)(Y,T),\bar{Z}) = \ a^r P_1(T)Q(X,\bar{Y},\bar{Z}),$ (2.10)a

or equivalently

$a^r(\nabla Q)(\bar{X},Y,\bar{Z},T) - Q(\bar{X},(\nabla F)(\bar{Y},T\ ),\bar{Z}) + a^r Q\big((\nabla F)(X,T),Y,\bar{Z}\big)$
$+a^r Q\big(\bar{X},Y,(\nabla F)(Z,T)\big) = \ a^r P_1(T)Q(\bar{X},Y,\bar{Z}),$ (2.10)b

or equivalently

$a^r(\nabla Q)(\bar{X},\bar{Y},Z,T) + a^r Q\big((\nabla F)(X,T),\bar{Y},Z\big) + a^r Q(\bar{X},(\nabla F)(Y,T),Z)$
$-Q(\bar{X},\bar{Y},(\nabla F)(\bar{Z},T)) = \ a^r P_1(T)Q(\bar{X},\bar{Y},Z).$ (2.10)c

Definition(2.11). A (1),(2),(3),(1,2),(1,3),(2,3) and (1,2,3)- recurrent H-HSU-structure manifold is said to be Q-symmetric or Ricc-symmetric,if

$P_1(T) = 0$ (2.11)

In the above equations,

Theorem (2.1) A Q-(1)-recurrent H-HSU-manifold is Q-(2)- recurrent for same recurrent parameter ,if

$Q((\nabla F)(\bar{X},\mathrm{T}),Y,Z) = Q(X,(\nabla F)(\bar{Y},\mathrm{T}),Z).$ (2.12)

Proof: if a Q-(1)-recurrent H-HSU-manifold is Q-(2)-recurrent ,then we have

$a^r(\nabla Q)(X,Y,Z,T) - Q((\nabla F)(\bar{X},T),Y,Z) - a^r P_1(T)Q(X,Y,Z) = a^r(\nabla Q)(X,Y,Z,T) - Q(X,(\nabla F)(\bar{Y},T),Z) - a^r P_1(T)Q(X,Y,Z)$ (2.13)

From equation(2.13),we have the equation (2.12).

Conversely, let the equation (2.12)is satisfied and the manifold is Q(1)-recurrent ,then using equation in (2.1),we get

$a^r(\nabla Q)(X,Y,Z,T) - Q(X,(\nabla F)(\bar{Y},T),Z) = a^r P_1(T)Q(X,Y,Z)$

Which shows that the manifold is Q-(2)-recurrent.

Note(2.1)-Similarly, it can be shown that the Q-(3)-recurrent H-HSU-structure manifold is Q-(1)-recurrent, if

$Q\big((\nabla F)(\bar{X},T),Y,Z\big) = Q\big(X,Y(\nabla F)(\bar{Z},T)\big),$

Or Q-(2)-recurrent, if

$Q(X,(\nabla F)(\bar{Y},T),Z) = Q\big(X,Y(\nabla F)(\bar{Z},T)\big),$

And Ricci-(1)-recurrent H-HSU-manifold is Ricci-(2)-recurrence, if

$Ric((\nabla F)(\bar{Y},T),Z) = Ric(Y,(\nabla F)(\bar{Z},T)$ for the same recurrence parameter.

Theorem (2.2) A Q-(1,2)-recurrent H-HSU-structure manifold is Q-(1,3)-recurrent for the same recurrence parameter , iff

$Q(\bar{X},(\nabla F)(\bar{Y},T),Z) = Q\big(\bar{X},Y(\nabla F)(\bar{Z},T)\big).$ (2.14)

Proof. Assuming that the Q-(1,2)-recurrent H-HSU-structure manifold is Q-(1,3)-recurrent then using equation(2.14) in equation (2.6)b, we get

$a^r(\nabla Q)(\bar{X},Y,Z,T) - Q\big(\bar{X},Y(\nabla F)(\bar{Z},T)\big) + a^r Q\big((\nabla F)(X,T),Y,Z\big)$
$= a^r P_1(T)Q(\bar{X},Y,Z),$

Which shows that the manifold is Q-(1,3)-recurrent.

Note(2.2).Similarly ,it can be shown that the Q-(2,3)-recurrent H-HSU-structure manifold is Q-(1,2)-recurrent, iff

$Q((\nabla F)(\bar{X},T),\bar{Y},Z) = Q(X,\bar{Y},(\nabla F)(\bar{Z},T))$

Or Q-(1,3)-recurrent,iff

Q(X,$(\nabla F)(\bar{Y},T),Z) = Q((\nabla F)(\bar{X},T),Y,\bar{Z})$

For the same recurrence parameter .

Theorem (2.3). A Q-(1,2)-recurrent H-HSU-structure manifold is Q-(1)-recurrent for the same recurrent parameter provided

$a^r Q(X,(\nabla F)(Y,T),Z) = 0$ (2.15)

Proof . Let the manifold is (1)-recurrent in Q then barring Y in equation (2.1),we get.

$$a^r(\nabla Q)(X,\bar{Y},Z,T) - Q((\nabla F)(\bar{X},T),\bar{Y},Z) = a^r P_1(T)Q(X,\bar{Y},Z) \qquad (2.16)$$

Now, assuming that a Q-(1,2)-recurrent H-HSU-structure-manifold is Q-(1)-recurrent then comparing ,equation (2.6)a and (2.16),we get the equation (2.15).

Note(1.3). Similarly, it can be shown that a Q-(1,2)-recurrent H-HSU-structure manifold is Q(2)-recurrent for the same recurrence parameter,provided.

$$a^r Q((\nabla F)(X,T),Y,Z) = 0. \qquad (2.17)$$

Remark(1.1). Theorems of the type(2.3) can also be proved taking Q-(1,3)or Q(2,3)-recurrent H-HSU-structure manifold instead of Q-(1,2)-recurrent and and Q-(1) orQ-(3) or Q-(2)-recurrent manifold in place of Q-(1)-recurrent manifold.

Theorem (2.4) A Q(1,2,3)-recurrent H-HSU-structure manifold is Q-(1)-recurrent for the same recurrent parameter provided.

$$a^r Q(X,(\nabla F)(Y,T),\bar{Z}) + a^r Q\big(X,\bar{Y},(\nabla F)(Z,T)\big) = 0\,. \qquad (2.18)$$

Proof. Let the manifold is Q-(1)-recurrent, then barring Y and Z in equation (2.1),we get

$$a^r(\nabla Q)(X,\bar{Y},\bar{Z},T) - Q((\nabla F)(\bar{X},T),\bar{Y},\bar{Z}) = a^r P_1(T)Q(X,\bar{Y},\bar{Z}) \qquad (2.19)$$

Now assuming that a Q-(1,2,3)-recurrent manifold is Q-(1)-recurrent, then comparing equations (2.10)a and (2.19),we get the equation (2.18).

Note(2.4). Theorems of the type (1.4) can also be proved taking Q-(2) or Q-(3)-recurrent H-HSU-structure manifold instead of Q-(1)-recurrent H-HSU-structure manifold.

Theorem(1.5) A Q-(1,2,3)-recurrent H-HSU-structure manifold is Q-(1,2)-recurrent H-HSU-manifold for the same recurrence parameter ,provided.

$$a^r Q(X,\bar{Y}(\nabla F)(Z,T) = 0 \qquad (2.20)$$

Proof. Let the manifold is Q-(1,2)-recurrent, then barring Z in equation (2.6)a ,we get

$$a^r(\nabla Q)(X,\bar{Y},\bar{Z},T) - Q\big((\nabla F)(\bar{X},T),\bar{Y},\bar{Z}\big) + a^r Q(X,(\nabla F)(Y,T),\bar{Z})$$
$$= a^r P_1(T)Q(X,\bar{Y},\bar{Z}), \qquad (2.21)$$

Now, assuming that a Q-(1,2,3)-recurrent H-HSU-structure manifold is Q-(1,2)-recurrent and then comparing equations (2.10)a and (2.21),we get the equation (2.20).

Note(1.5). Theorems of the type(2.5)can also be proved taking Q-(1,3)-or Q-(2,3)-recurrent H-HSU-structure manifold instead of Q-(1,2)-recurrent H-HSU-structure manifold.

Theorem(2.6). In a recurrent H-HSU-structure manifold, if any two of following conditions hold for the same recurrence parameter, then the third also hold:

(a) It is conformal (1)-recurrent ,
(b) It is conharmonic (1)-recurrent,
(c) It is concircular (1)-recurrent.

Proof. From the equation (1.6),(1.7),(1.8) we have

$$C(X,Y,Z) = L(X,Y,Z) + \frac{n}{(n-2)}[K(X,Y,Z) - V(X,Y,Z)] \qquad (2.22)$$

Barring X in equations (2.22) ,we get

$$C(\bar{X},Y,Z) = L(\bar{X},Y,Z) + \frac{n}{(n-2)}[K(\bar{X},Y,Z) - V(\bar{X},Y,Z)]. \qquad (2.23)$$

Now, from equation from from (1.1) and (2.23),we have

$$a^r P_1(T)C(X,Y,Z) = a^r P_1(T)L(X,Y,Z) + \frac{na^r}{(n-2)}P_1(T)\{K(X,Y,Z) - V(X,Y,Z)\} \qquad (2.24)$$

Differentiating equation(2.23) with respect to T, using equation (2.23) and then barring X in the resulting equation, we get

$$-a^r(\nabla C)(X,Y,Z,T) + C((\nabla F)(\bar{X},T),Y,Z) = -a^r((\nabla L)(X,Y,Z,T) + L((\nabla F)(\bar{X},T),Y,Z) + \frac{n}{(n-1)}\{-a^r(\nabla K)(X,Y,Z,T) + K((\nabla F)(\bar{X},T),Y,Z) + a^r(\nabla V)(X,Y,Z,T) - V((\nabla F)(X,T),Y,Z)\}. \qquad (2.25)$$

Adding equation (2.24) and (2.25), we get

$$-a^r(\nabla C)(X,Y,Z,T) + C((\nabla F)(\bar{X},T),Y,Z) + a^r P_1(T)C(X,Y,Z) = -a^r((\nabla L)(X,Y,Z,T) + L((\nabla F)(\bar{X},T),Y,Z) + a^r P_1(T)L(X,Y,Z) + \frac{n}{(n-2)}\{-a^2(\nabla K)(X,Y,Z,T) + K((\nabla F)(\bar{X},T),Y,Z) + a^r P_1(T)K(X,Y,Z) + a^r(\nabla V)(X,Y,Z,T) - V\big((\nabla F)(X,T),Y,Z\big) - a^r P_1(T)K(X,Y,Z)\} \qquad (2.26)$$

If a (1)-recurrent H-HSU-Structure manifold is conformal-(1) recurrent and conharmonic-(1)-recurrent for the same recurrence parameter then from equation (2.16), we get

$$a^r(\nabla V)(X,Y,Z,T) - V\big((\nabla F)(\bar{X},T),Y,Z\big) = a^r P_1(T)V(X,Y,Z),$$

Which shows that the manifolds is concircular –(1)-recurrent.

Similarly, it can be shown that if the recurrent manifold is either conformal-(1)-recurrent and concircular-(1)-recurrent or conharmonic-(1)-recurrent and concircular-(1)-recurrent then it is either conharmonic-(1)-recurrent or conformal-(1)-recurrent for same recurrence parameter.

Theorem(2.7) In a (1,2) recurrent H-HSU-structure manifold, if any two of following conditions hold for the same recurrence parameter, then the third also hold:

(a) It is conformal (1,2)-recurrent ,
(b) It is conharmonic (1,2)-recurrent,
(c) It is concircular (1,2)-recurrent.

Barring X and Y in equations (2.22) ,we get

$$C(\bar{X},\bar{Y},Z) = L(\bar{X},\bar{Y},Z) + \frac{n}{(n-2)}[K(\bar{X},\bar{Y},Z) - V(\bar{X},\bar{Y},Z)]. \quad (2.27)$$

Now, from equation (1.1) and (2.27),we have

$$a^r P_1(T)C(X,\bar{Y},Z) = a^r P_1(T)L(X,\bar{Y},Z) + \frac{na^r}{(n-2)}P_1(T)\{K(X,\bar{Y},Z) - V(X,\bar{Y},Z)\} \quad (2.28)$$

Differentiating equation(2.27) with respect to T, using equation (2.27) and then barring X in the resulting equation, we get

$$-a^r(\nabla C)(X,\bar{Y},Z,T) + C\big((\nabla F)(\bar{X},T),\bar{Y},Z\big) - a^r C(X,(\nabla F)(Y,T),Z)$$
$$= -a^r((\nabla L)(X,\bar{Y},Z,T) \quad + L((\nabla F)(\bar{X},T),\bar{Y},Z) - a^r L(X,(\nabla F)(Y,T),Z)$$
$$+\frac{n}{(n-1)}\{-a^r(\nabla K)(X,\bar{Y},Z,T) + K((\nabla F)(\bar{X},T),\bar{Y},Z) - a^r K(X,(\nabla F)(Y,T),Z)$$
$$+a^r(\nabla V)(X,\bar{Y},Z,T) - V((\nabla F)(\bar{X},T),\bar{Y},Z)\} + a^r V(X,(\nabla F)(Y,T),Z)\}. \quad (2.29)$$

Adding equations (2.28) and (2.29),we get

$$-a^r(\nabla C)(X,\bar{Y},Z,T) + C\big((\nabla F)(\bar{X},T),\bar{Y},Z\big) - a^r C(X,(\nabla F)(Y,T),Z) + a^r P_1(T)C(X,\bar{Y},Z)$$
$$= -a^r((\nabla L)(X,\bar{Y},Z,T) \; + L((\nabla F)(\bar{X},T),\bar{Y},Z) - a^r L(X,(\nabla F)(Y,T),Z)$$
$$+ a^r P_1(T)L(X,\bar{Y},Z) \quad +$$
$$\frac{n}{(n-2)}\{-a^r(\nabla K)(X,\bar{Y},Z,T) + K((\nabla F)(\bar{X},T),\bar{Y},Z)$$
$$-a^r K(X,(\nabla F)(Y,T),Z) + a^r P_1(T)K(X,\bar{Y},Z) + a^r(\nabla V)(X,\bar{Y},Z,T)$$
$$-V((\nabla F)(\bar{X},T),\bar{Y},Z)\} + a^r V(X,(\nabla F)(Y,T),Z) - a^r P_1(T)V(X,\bar{Y},Z)\} \quad (2.30)$$

If a (1,2)-recurrent H-HSU-Structure manifold is conformal-(1,2) recurrent and conharmonic-(1,2)-recurrent for the same recurrence parameter then from equation (2.30), we get

$$a^r(\nabla V)(X,\bar{Y},Z,T) - V\big((\nabla F)(\bar{X},T),\bar{Y},Z\big) + a^r V(X,(\nabla F)(Y,T),Z) = a^r P_1(T)V(X,\bar{Y},Z).$$

Which shows that the manifolds is concircular –(1,2)-recurrent.

Similarly, it can be shown that if the recurrent manifold is either conformal-(1,2)-recurrent and concircular-(1,2)-recurrent or conharmonic-(1,2)-recurrent and concircular-(1,2)-recurrent then it is either conharmonic-(1,2)-recurrent or conformal-(1,2)-recurrent for same recurrence parameter.

Theorem (2.8)) In a (1,2,3) recurrent H-HSU-structure manifold, if any two of following conditions hold for the same recurrence parameter, then the third also hold:

(a) It is conformal (1,2,3)-recurrent ,
(b) It is conharmonic (1,2,3)-recurrent,
(c) It is concircular (1,2,3)-recurrent.

Barring X,Y and Z in equations (2.22) ,we get

$$C(\bar{X},\bar{Y},\bar{Z}) = L(\bar{X},\bar{Y},\bar{Z}) + \frac{n}{(n-2)}[K(\bar{X},\bar{Y},\bar{Z}) - V(\bar{X},\bar{Y},\bar{Z})]. \quad (2.31)$$

Now, from equation (1.1) and (2.31),we have

$$a^r P_1(T)C(X,\bar{Y},Z) = a^r P_1(T)L(X,\bar{Y},Z) + \frac{na^r}{(n-2)}P_1(T)\{K(X,\bar{Y},Z) - V(X,\bar{Y},Z)\} \quad (2.32)$$

Differentiating equation(2.31) with respect to T, using equation (2.31) and then barring X in the resulting equation, we get

$$-a^r(\nabla C)(X,\bar{Y},\bar{Z},T) + C\big((\nabla F)(\bar{X},T),\bar{Y},\bar{Z}\big) - a^r C(X,(\nabla F)(Y,T),\bar{Z}) - a^r C(X,\bar{Y},(\nabla F)(Z,T)$$
$$= a^r((\nabla L)(X,\bar{Y},\bar{Z},T) + L\big((\nabla F)(\bar{X},T),\bar{Y},\bar{Z}\big) - a^r L(X,(\nabla F)(Y,T),\bar{Z})$$
$$-a^r L(X,\bar{Y},(\nabla F)(Z,T)) + \frac{n}{(n-1)}\{-a^r(\nabla K)(X,\bar{Y},\bar{Z},T) + K((\nabla F)(\bar{X},T),\bar{Y},\bar{Z})$$
$$-a^r K(X,(\nabla F)(Y,T),\bar{Z}) - \big(X,\bar{Y}(\nabla F)(Z,T) + a^r(\nabla V)(X,\bar{Y},\bar{Z},T) - V\big((\nabla F)(\bar{X},T),\bar{Y},\bar{Z}\big)\}$$
$$+a^r V(X,(\nabla F)(Y,T),\bar{Z}) + a^r V(X,\bar{Y}(\nabla F)(Z,T))\}. \quad (2.33)$$

Adding equations (2.32) and (2.33),we get

$$-a^r(\nabla C)(X,\bar{Y},\bar{Z},T) + C\big((\nabla F)(\bar{X},T),\bar{Y},\bar{Z}\big) - a^r C(X,(\nabla F)(Y,T),\bar{Z})$$
$$-a^r C\big(X,\bar{Y}(\nabla F)(Z,T)\big) + a^r P_1(T)C(X,\bar{Y},\bar{Z})$$
$$= -a^r((\nabla L)(X,\bar{Y},\bar{Z},T) + L((\nabla F)(\bar{X},T),\bar{Y},\bar{Z}) - a^r L(X,(\nabla F)(Y,T),\bar{Z})$$

$$-a^r L(X,\bar{Y}(\nabla F)(Z,T)) + a^r P_1(T)L(X,\bar{Y},\bar{Z})$$
$$+ \frac{n}{(n-1)}\{-a^r(\nabla K)(X,\bar{Y},\bar{Z},T)$$
$$+K((\nabla F)(\bar{X},T),\bar{Y},\bar{Z}) - a^r K(X,(\nabla F)(Y,T),\bar{Z}) - a^r K\big(X,\bar{Y}(\nabla F)(Z,T)\big)$$
$$+a^r P_1(T)K(X,\bar{Y},\bar{Z}) + a^r(\nabla V)(X,\bar{Y},\bar{Z},T) - V((\nabla F)(\bar{X},T),\bar{Y},\bar{Z})\}$$

$+a^r V(X,(\nabla F)(Y,T),\bar{Z}) - a^r P_1(T)V(X,\bar{Y},\bar{Z})\}$ (2.34)

If a (1,2,3)-recurrent H-HSU-Structure manifold is conformal-(1,2,3) recurrent and conharmonic-(1,2,3)-recurrent for the same recurrence parameter then from equation (2.34), we get

$$a^r(\nabla V)(X,\bar{Y},\bar{Z},T) - V\big((\nabla F)(\bar{X},T),\bar{Y},\bar{Z}\big) + a^r V(X,(\nabla F)(Y,T),\bar{Z}) + a^r V\big(X,\bar{Y}(\nabla F)(Z,T)\big) = a^r P_1(T)V(X,\bar{Y},\bar{Z}).$$

Which shows that the manifolds is concircular –(1,2,3)-recurrent.
Similarly, it can be shown that if the recurrent manifold is either conformal-(1,2,3)-recurrent and concircular-(1,2,3)-recurrent or conharmonic-(1,2,3)-recurrent and concircular-(1,2,3)-recurrent then it is either conharmonic-(1,2,3)-recurrent or conformal-(1,2,3)-recurrent for same recurrence parameter.

References

[1] C.J. Hsu, "On some structure which are similar to quaternion structure", *Tohoku Math. J.*, (12), 1960, pp. 403-428.
[2] Q. Khan, "On recurrent Riemannian manifolds", *Kuungpook Math. J.*, (44), 2004, pp. 269-276.
[3] R.S. Mishra, S.B. Pandey and U. Sharma "Pseudo n-Recurrent Manifold with KH-Structure", *Indian J. pure applied Mathematics*, 7(10), 1974, pp. 1277-1287.
[4] R.S. Mishra, "A course in tensors with applications to Riemannian Geometry", II edition Pothishala Pvt., Ltd. Allahabad (1973).
[5] R.S. Mishra, "Structure on a differentiable manifold and their applications", Chandrama Prakashan, 50-A, Balrampur House, Allahabad (1984).
[6] S.B. Pandey and Lata Dasila," On General Differentiable Manifold". *U. Scientist Phyl. Sciences*, Vol.7, No. 2,147-152(1995).
[7] S.B. Pandey and Lata Dasila, "On Birecurrent General Differentiable Manifold". *U. Scientist Phyl. Sciences*, Vol.9, No.1, 1997.

On Hyperbolic Hsu-Structure Manifold,BIRecurrent and Symmetry

1.Introduction

If a differentiable manifold Vn, of differentiability class C^∞.there be in Vn,a vector valued linear function F of classC^∞, satisfying the algebraic equation

$\bar{\bar{X}}$ = - a$^r X$, for arbitrary vector field X . (1.1)

Where $\bar{X}$= FX , 0≤ $r \leq n$ and 'a'is real or imaginary number,then{F} is said to give to Vn a Hyperbolic HSU-structure defined by the equations(1.1) and the manifold Vn is called a Hyperbolic HSU –structure manifold.Hyperbolic HSU-structure manifold or briefly H-Hsu-structure manifold.

Remark(1.1) : The equation (1.1) gives different structures for different values of ‘a’ and r .

If a= ± 1 and r =2 ,it is an almost complex structure.
If a= $\pm i$ and r = 2 , it is an almost product structure or a hyperbolic almost complex structure.
If a = 0, it an almost tangent structure or almost hyperbolic tangent structure.
If a≠0, s It is the hyperbolic π-structure.

Let the Hsu – structure Vn, be endowed with a Hermitian metric tensor g, such that
g($\bar{X}$,$\bar{Y}$) –a^r g(X,Y) =0,
Then{F,g} is said to give Vn a hyperbolic Hsu-structure metric manifold.
Agreement (1.1):In what follows and the above, the equations containing X,Y,Z…………,etc. hold for these arbitrary vector in Vn,
The curvature tensor K,a vector –valued tri-linear function w.r.t the connexion D is given by

$$K(X,Y,Z) = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z, \quad (1.2)a$$

Where

$$[X,Y] = D_X Y - D_Y X \quad (1.2)b$$

The Ricci tensor in $V_n$is given by

$$\text{Ric}(Y,Z) = (C_1^1 K)(Y,Z). \quad (1.3)$$

Where by ($C_1^1 K)(Y,Z)$, we mean the contraction of $K(X,Y,Z)$with respect the first slot.
For Ricci tensor, we also have

$$Ric(Y,Z) = Ric(Z,Y), \quad (1.4)a$$

$$Ric(Y,Z) = g(r(Y),Z) = g\big(Y,r(Z)\big), \quad (1.4)b$$

$$(C_1^1 r) = R \quad (1.4)c$$

Let W,C,L and V be the Projective,Conformal, conharmonic and concircular curvature tensors respectively given by

$$W(X,Y,Z) = K(X,Y,Z) - \frac{1}{(n-1)}[Ric(Y,Z)X - Ric(X,Z)Y] \quad (1.5)$$

$$C(X,Y,Z) = -\frac{1}{(n-2)}\{Ric(Y,Z)X - Ric(X,Z)Y - g(X,Z)r(Y) + g(Y,Z)r(X)\} + \frac{R}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]. \quad (1.6)$$

$$L(X,Y,Z) = K(X,Y,Z) - \frac{1}{(n-2)}[Ric(Y,Z)X - Ric(X,Z)Y - g(X,Z)r(Y) + g(Y,Z)r(X)]. \quad (1.7)$$

$$V(X,Y,Z) = K(X,Y,Z) - \frac{R}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \quad (1.8)$$

A manifold is said to be recurrent, if

$$(\nabla K)(X,Y,Z,T) = A(T_1)K(X,Y,Z).$$

The recurrent manifold is said to be symmetric, if
$A(T_{1)} = 0$, in the equation (1.9).

II BIRECURRENCE AND SYMMETERY OF DIFFERENT KINDS

Let Q, a vector – valued trilinear function be any one of the curvature tensors K,W,C,L or V. Then we will define recurrence of different kinds as follows:

Definition(2.1). A –HSU-structure manifold is said to be (1)-birecurrent in Q,if

$$a^r(\nabla\nabla Q)(X,Y,Z,T,S) - (\nabla Q)\big((\nabla F)(\bar{X},T),Y,Z,S\big) - (\nabla Q)\big((\nabla F)(\bar{X},S),Y,Z,T\big) - Q((\nabla\nabla F)(\bar{X},T,S),Y,Z) = a^r P_1(T,S)Q(X,Y,Z) \quad (2.1)$$

Where P_2(T,S) is non – vanishing C^∞,called birecurrence parameter.

Definition(2.2). A H-HSU- structure manifold is said to be Ricci (1)-birecurrent ,if

$$a^r(\nabla\nabla \mathrm{Ric})(Y,Z,T,S) - (\nabla \mathrm{Ric})(\big((\nabla F)(\bar{Y},T),Z,S\big) - (\nabla Ric)\big((\nabla F)(\bar{Y},S),Z,T\ \big) - Q((\nabla\nabla Ric)(\bar{Y},T,S),Z) = a^r P_2(T,S)Ric(Y,Z). \quad (2.2)$$

Definition(2.3). A H-HSU- structure manifold is said to be- (1,2)-birecurrent inQ,if

$$a^r(\nabla\nabla Q)(X,\bar{Y},Z,T,S) - (\nabla Q)\big((\nabla F)(\bar{X},T),\bar{Y},Z,S\big) - (\nabla Q)\big((\nabla F)(\bar{X},S),\bar{Y},Z,T\big) + a^r(\nabla Q)(X,(\nabla F)(Y,T),Z,S) + a^r(\nabla Q)(X,(\nabla F)(Y,S),Z,T) - Q\big((\nabla F)(\bar{X},T),(\nabla F)(Y,S),Z\big) - Q\big((\nabla F)(\bar{X},S),(\nabla F)Y,T\big),Z) - \mathrm{Q}\big((\nabla\nabla \mathrm{F})(\bar{\mathrm{X}},T,S),\bar{Y},Z\big) + a^r Q(X,(\nabla\nabla F)(Y,T,S),Z) = a^r P_2(T,S)Q(X,\bar{Y},Z), \quad (2.3)$$

Definition(2.4). A H-HSU- structure manifold is said to be Ricci- (1)- recurrent ,if

$$a^r(\nabla \mathrm{Ric})(,Y,Z,T) - Ric((\nabla F)(\bar{Y},T),Z) = a^r P_1(T)Ric(Y,Z), \quad (2.4)$$

Definition(2.5). A H-HSU- structure manifold is said to be Ricci- (2)- recurrent ,if

$$a^r(\nabla \mathrm{Ric})(,Y,Z,T) - Ric(Y,(\nabla F)(\bar{Z},T) = a^r P_1(T)Ric(Y,Z), \quad (2.5)$$

Definition(2.6). A –HSU-structure manifold is said to be (1,2)-recurrent in Q,if

$$a^r(\nabla Q)(X,\bar{Y},Z,T) - Q\big((\nabla F)(\bar{X},T),\bar{Y},Z\big) + a^r Q(X,(\nabla F)(Y,T),Z)$$

$= a^r P_1(T) Q(X, \bar{Y}, Z),$ (2.6)a

or equivalently

$a^r (\nabla Q)(\bar{X}, Y, Z, T) - Q(\bar{X}(\nabla F)(\bar{Y}, T), Z) + a^r Q\big((\nabla F)(X, T), Y, Z\big)$

$= a^r P_1(T) Q(\bar{X}, Y, Z),$ (2.6)b

or equivalently

$a^{2r} (\nabla Q)(X, Y, Z, T) - a^r (Q\big((\nabla F)(\bar{X}, T), Y, Z\big) - a^r Q(X, (\nabla F)(\bar{Y}, T), Z)$

$= a^{2r} P_1(T) Q(X, Y, Z),$ (2.6)c

Definition (2.7). A –HSU-structure manifold is said to be (1,2)-recurrent in Q,if

$a^r (\nabla Q)(X, Y, \bar{Z}, T) - Q\big((\nabla F)(\bar{X}, T), Y, \bar{Z}\big) + a^r Q\big(X, Y, (\nabla F)(Z, T)\big)$

$= a^r P_1(T) Q(X, Y, \bar{Z}),$ (2.7)a

or equivalently

$a^r (\nabla Q)(\bar{X}, Y, Z, T) - Q(\bar{X}, Y(\nabla F)(\bar{Z}, T\ )) + a^r Q\big((\nabla F)(X, T), Y, Z\big)$

$= a^r P_1(T) Q(\bar{X}, Y, Z),$ (2.7)b

or equivalently

$a^{2r} (\nabla Q)(X, Y, Z, T) - a^r (Q\big((\nabla F)(\bar{X}, T), Y, Z\big) - a^r Q\big(X, Y(\nabla F)(\bar{Z}, T)\big)$

$= a^{2r} P_1(T) Q(X, Y, Z),$ (2.7)c

Definition(2.8). A –HSU-structure manifold is said to be (2,3)-recurrent in Q,if

$a^r (\nabla Q)(X, Y, \bar{Z}, T) - Q(X, (\nabla F)(\bar{Y}, T), \bar{Z}) + a^r Q\big(X, Y, (\nabla F)(Z, T)\big)$

$= a^r P_1(T) Q(X, Y, \bar{Z}),$ (2.8)a

or equivalently

$a^r (\nabla Q)(X, \bar{Y}, Z, T) - Q(X, \bar{Y}, (\nabla F)(\bar{Z}, T\ ) + a^r Q(X, (\nabla F)(Y, T), Z)$

$= a^r P_1(T) Q(X, \bar{Y}, Z),$ (2.8)b

or equivalently

$a^{2r} (\nabla Q)(X, Y, Z, T) - a^r (Q(X, (\nabla F)(\bar{Y}, T), Z) - a^r Q(X, (\nabla F)(Y, T), \bar{Z})$

$= a^{2r} P_1(T) Q(X, Y, Z).$ (2.8)c

Definition(2.9). A H-HSU- structure manifold is said to be Ricci- (1,2)- recurrent ,if

$a^r (\nabla \mathrm{Ric})(, Y, \bar{Z}, T) - Ric\big((\nabla F)(\bar{Y}, T), Z\big) + a^r Ric(Y, (\nabla F), Z, T))$

$= a^r P_1(T) Ric(Y, \bar{Z})$, (2.9)a

or equivalently

$a^r (\nabla \mathrm{Ric})(, \bar{Y}, Z, T) - Ric\big(\bar{Y}, (\nabla F)(\bar{Z}, T)\big) + a^r Ric((\nabla F)(Y, T), Z)$

$= a^r P_1(T) Ric(\bar{Y}, Z),$ (2.9)b

Or equivalently

$a^{2r}(\nabla \mathrm{Ric})(Y,Z,T) - a^{r}(Ric((\nabla F)(\bar{Y},T),Z) - a^{r}Ric(\bar{Y},(\nabla F)(Z,T)$
$= a^{2r}P_1(T)Ric(Y,Z).$ (2.9)c

Definition(2.10). A –HSU-structure manifold is said to be (1,2,3)-recurrent if
$a^{r}(\nabla Q)(X,\bar{Y},\bar{Z},T) - Q((\nabla F)(\bar{X},T),\bar{Y},\bar{Z}) + a^{r}Q(X,\bar{Y},(\nabla F)(Z,T))$
$+a^{r}Q(X,(\nabla F)(Y,T),\bar{Z}) = a^{r}P_1(T)Q(X,\bar{Y},\bar{Z}),$ (2.10)a

or equivalently
$a^{r}(\nabla Q)(\bar{X},Y,\bar{Z},T) - Q(\bar{X},(\nabla F)(\bar{Y},T\ ),\bar{Z}) + a^{r}Q((\nabla F)(X,T),Y,\bar{Z})$
$+a^{r}Q(\bar{X},Y,(\nabla F)(Z,T)) = a^{r}P_1(T)Q(\bar{X},Y,\bar{Z}),$ (2.10)b

or equivalently
$a^{r}(\nabla Q)(\bar{X},\bar{Y},Z,T) + a^{r}Q((\nabla F)(X,T),\bar{Y},Z) + a^{r}Q(\bar{X},(\nabla F)(Y,T),Z)$
$-Q(\bar{X},\bar{Y},(\nabla F)(\bar{Z},T)) = a^{r}P_1(T)Q(\bar{X},\bar{Y},Z).$ (2.10)c

Definition(2.11). A (1),(2),(3),(1,2),(1,3),(2,3) and (1,2,3)- recurrent H-HSU-structure manifold is said to be Q-symmetric or Ricc-symmetric,if
$P_1(T) = 0$ (2.11)
In the above equations,
Theorem (2.1) A Q-(1)-recurrent H-HSU-manifold is Q-(2)- recurrent for same recurrent parameter ,if

$Q((\nabla F)(\bar{X},T),Y,Z) = Q(X,(\nabla F)(\bar{Y},T),Z).$ (2.12)

Proof: if a Q-(1)-recurrent H-HSU-manifold is Q-(2)-recurrent ,then we have

$a^{r}(\nabla Q)(X,Y,Z,T) - Q((\nabla F)(\bar{X},T),Y,Z) - a^{r}P_1(T)Q(X,Y,Z) = a^{r}(\nabla Q)(X,Y,Z,T) - Q(X,(\nabla F)(\bar{Y},T),Z) - a^{r}P_1(T)Q(X,Y,Z)$ (2.13)

From equation(2.13),we have the equation (2.12).
Conversely, let the equation (2.12)is satisfied and the manifold is Q(1)-recurrent ,then using equation in (2.1),we get

$a^{r}(\nabla Q)(X,Y,Z,T) - Q(X,(\nabla F)(\bar{Y},T),Z) = a^{r}P_1(T)Q(X,Y,Z)$
Which shows that the manifold is Q-(2)-recurrent.
Note(2.1)-Similarly, it can be shown that the Q-(3)-recurrent H-HSU-structure manifold is Q-(1)-recurrent, if
$Q((\nabla F)(\bar{X},T),Y,Z) = Q(X,Y(\nabla F)(\bar{Z},T)),$
Or Q-(2)-recurrent, if

$Q(X,(\nabla F)(\bar{Y},T),Z) = Q(X,Y(\nabla F)(\bar{Z},T)),$
And Ricci-(1)-recurrent H-HSU-manifold is Ricci-(2)-recurrence, if

$Ric((\nabla F)(\bar{Y},T),Z) = Ric(Y,(\nabla F)(\bar{Z},T)$ for the same recurrence parameter.

Theorem (2.2) A Q-(1,2)-recurrent H-HSU-structure manifold is Q-(1,3)-recurrent for the same recurrence parameter , iff

$Q(\bar{X},(\nabla F)(\bar{Y},T),Z) = Q\big(\bar{X},Y(\nabla F)(\bar{Z},T)\big).$ (2.14)

Proof. Assuming that the Q-(1,2)-recurrent H-HSU-structure manifold is Q-(1,3)-recurrent then using equation(2.14) in equation (2.6)b, we get

$a^r(\nabla Q)(\bar{X},Y,Z,T) - Q\big(\bar{X},Y(\nabla F)(\bar{Z},T)\big) + a^r Q\big((\nabla F)(X,T),Y,Z\big)$
$= a^r P_1(T)Q(\bar{X},Y,Z),$

Which shows that the manifold is Q-(1,3)-recurrent.

Note(2.2).Similarly ,it can be shown that the Q-(2,3)-recurrent H-HSU-structure manifold is Q-(1,2)-recurrent, iff

$Q((\nabla F)(\bar{X},T),\bar{Y},Z) = Q(X,\bar{Y},(\nabla F)(\bar{Z},T))$

Or Q-(1,3)-recurrent,iff

Q(X,$(\nabla F)(\bar{Y},T),Z) = Q((\nabla F)(\bar{X},T),Y,\bar{Z})$

For the same recurrence parameter .

Theorem (2.3). A Q-(1,2)-recurrent H-HSU-structure manifold is Q-(1)-recurrent for the same recurrent parameter provided

$a^r Q(X,(\nabla F)(Y,T),Z) = 0$ (2.15)

Proof . Let the manifold is (1)-recurrent in Q then barring Y in equation (2.1),we get.

$a^r(\nabla Q)(X,\bar{Y},Z,T) - Q((\nabla F)(\bar{X},T),\bar{Y},Z) = a^r P_1(T)Q(X,\bar{Y},Z)$ (2.16)

Now, assuming that a Q-(1,2)-recurrent H-HSU-structure-manifold is Q-(1)-recurrent then comparing ,equation (2.6)a and (2.16),we get the equation (2.15).

Note(1.3). Similarly, it can be shown that a Q-(1,2)-recurrent H-HSU-structure manifold is Q(2)-recurrent for the same recurrence parameter,provided.

$a^r Q((\nabla F)(X,T),Y,Z) = 0.$ (2.17)

Remark(1.1). Theorems of the type(2.3) can also be proved taking Q-(1,3)or Q(2,3)-recurrent H-HSU-structure manifold instead of Q-(1,2)-recurrent and and Q-(1) orQ-(3) or Q-(2)-recurrent manifold in place of Q-(1)-recurrent manifold.

Theorem (2.4) A Q(1,2,3)-recurrent H-HSU-structure manifold is Q-(1)-recurrent for the same recurrent parameter provided.

$a^r Q(X,(\nabla F)(Y,T),\bar{Z}) + a^r Q\big(X,\bar{Y},(\nabla F)(Z,T)\big) = 0\,.$ (2.18)

Proof. Let the manifold is Q-(1)-recurrent, then barring Y and Z in equation (2.1),we get

$a^r(\nabla Q)(X,\bar{Y},\bar{Z},T) - Q((\nabla F)(\bar{X},T),\bar{Y},\bar{Z}) = a^r P_1(T)Q(X,\bar{Y},\bar{Z})$ (2.19)

Now assuming that a Q-(1,2,3)-recurrent manifold is Q-(1)-recurrent, then comparing equations (2.10)a and (2.19),we get the equation (2.18).

Note(2.4). Theorems of the type (1.4) can also be proved taking Q-(2) or Q-(3)-recurrent H-HSU-structure manifold instead of Q-(1)-recurrent H-HSU-structure manifold.

Theorem(1.5) A Q-(1,2,3)-recurrent H-HSU-structure manifold is Q-(1,2)-recurrent H-HSU-manifold for the same recurrence parameter ,provided.

$a^r Q(X,\bar{Y}(\nabla F)(Z,T) = 0$ (2.20)

Proof. Let the manifold is Q-(1,2)-recurrent, then barring Z in equation (2.6)a ,we get

$a^r(\nabla Q)(X,\bar{Y},\bar{Z},T) - Q\big((\nabla F)(\bar{X},T),\bar{Y},\bar{Z}\big) + a^r Q(X,(\nabla F)(Y,T),\bar{Z})$
$= a^r P_1(T)Q(X,\bar{Y},\bar{Z}),$ (2.21)

Now, assuming that a Q-(1,2,3)-recurrent H-HSU-structure manifold is Q-(1,2)-recurrent and then comparing equations (2.10)a and (2.21),we get the equation (2.20).

Note(1.5). Theorems of the type(2.5)can also be proved taking Q-(1,3)-or Q-(2,3)-recurrent H-HSU-structure manifold instead of Q-(1,2)-recurrent H-HSU-structure manifold.

Theorem(2.6). In a recurrent H-HSU-structure manifold, if any two of following conditions hold for the same recurrence parameter, then the third also hold:

(d) It is conformal (1)-recurrent ,
(e) It is conharmonic (1)-recurrent,
(f) It is concircular (1)-recurrent.

Proof. From the equation (1.6),(1.7),(1.8) we have

$C(X,Y,Z) = L(X,Y,Z) + \frac{n}{(n-2)}[K(X,Y,Z) - V(X,Y,Z)]$ (2.22)

Barring X in equations (2.22) ,we get

$C(\bar{X},Y,Z) = L(\bar{X},Y,Z) + \frac{n}{(n-2)}[K(\bar{X},Y,Z) - V(\bar{X},Y,Z)].$ (2.23)

Now, from equation from from (1.1) and (2.23),we have

$$a^r P_1(T)C(X,Y,Z) = a^r P_1(T)L(X,Y,Z) + \frac{na^r}{(n-2)} P_1(T)\{K(X,Y,Z) - V(X,Y,Z)\}$$

(2.24)

Differentiating equation(2.23) with respect to T, using equation (2.23) and then barring X in the resulting equation, we get

$$-a^r(\nabla C)(X,Y,Z,T) + C((\nabla F)(\bar{X},T),Y,Z) = -a^r((\nabla L)(X,Y,Z,T)$$
$$+L((\nabla F)(\bar{X},T),Y,Z) + \frac{n}{(n-1)}\{-a^r(\nabla K)(X,Y,Z,T)$$
$$+K((\nabla F)(\bar{X},T),Y,Z) + a^r(\nabla V)(X,Y,Z,T)$$
$$-V((\nabla F)(X,T),Y,Z)\}. \quad (2.25)$$

Adding equation (2.24) and (2.25), we get

$$-a^r(\nabla C)(X,Y,Z,T) + C((\nabla F)(\bar{X},T),Y,Z) + a^r P_1(T)C(X,Y,Z)$$
$$= -a^r((\nabla L)(X,Y,Z,T) + L((\nabla F)(\bar{X},T),Y,Z) + a^r P_1(T)L(X,Y,Z)$$
$$+\frac{n}{(n-2)}\{-a^2(\nabla K)(X,Y,Z,T) + K((\nabla F)(\bar{X},T),Y,Z) + a^r P_1(T)K(X,Y,Z)$$
$$+a^r(\nabla V)(X,Y,Z,T) - V\big((\nabla F)(X,T),Y,Z\big) - a^r P_1(T)K(X,Y,Z)\}$$

(2.26)

If a (1)-recurrent H-HSU-Structure manifold is conformal-(1) recurrent and conharmonic-(1)-recurrent for the same recurrence parameter then from equation (2.16), we get

$$a^r(\nabla V)(X,Y,Z,T) - V\big((\nabla F)(\bar{X},T),Y,Z\big) = a^r P_1(T)V(X,Y,Z),$$

Which shows that the manifolds is concircular –(1)-recurrent.
Similarly, it can be shown that if the recurrent manifold is either conformal-(1)-recurrent and concircular-(1)-recurrent or conharmonic-(1)-recurrent and concircular-(1)-recurrent then it is either conharmonic-(1)-recurrent or conformal-(1)-recurrent for same recurrence parameter.

Theorem(2.7) In a (1,2) recurrent H-HSU-structure manifold, if any two of following conditions hold for the same recurrence parameter, then the third also hold:

(c) It is conformal (1,2)-recurrent ,
(d) It is conharmonic (1,2)-recurrent,
(c) It is concircular (1,2)-recurrent.

Barring X and Y in equations (2.22) ,we get

$$C(\bar{X},\bar{Y},Z) = L(\bar{X},\bar{Y},Z) + \frac{n}{(n-2)}[K(\bar{X},\bar{Y},Z) - V(\bar{X},\bar{Y},Z)]. \quad (2.27)$$

Now, from equation (1.1) and (2.27),we have

$$a^r P_1(T)C(X,\bar{Y},Z) = a^r P_1(T)L(X,\bar{Y},Z) + \frac{na^r}{(n-2)} P_1(T)\{K(X,\bar{Y},Z) - V(X,\bar{Y},Z)\} \quad (2.28)$$

Differentiating equation(2.27) with respect to T, using equation (2.27) and then barring X in the resulting equation, we get

$$-a^r(\nabla C)(X,\bar{Y},Z,T) + C\big((\nabla F)(\bar{X},T),\bar{Y},Z\big) - a^r C(X,(\nabla F)(Y,T),Z)$$
$$= -a^r((\nabla L)(X,\bar{Y},Z,T) + L((\nabla F)(\bar{X},T),\bar{Y},Z) - a^r L(X,(\nabla F)(Y,T),Z)$$
$$+\frac{n}{(n-1)}\{-a^r(\nabla K)(X,\bar{Y},Z,T) + K((\nabla F)(\bar{X},T),\bar{Y},Z) - a^r K(X,(\nabla F)(Y,T),Z)$$
$$+a^r(\nabla V)(X,\bar{Y},Z,T) - V((\nabla F)(\bar{X},T),\bar{Y},Z)\} + a^r V(X,(\nabla F)(Y,T),Z)\}. \quad (2.29)$$

Adding equations (2.28) and (2.29),we get

$$-a^r(\nabla C)(X,\bar{Y},Z,T) + C\big((\nabla F)(\bar{X},T),\bar{Y},Z\big) - a^r C(X,(\nabla F)(Y,T),Z) + a^r P_1(T)C(X,\bar{Y},Z)$$
$$= -a^r((\nabla L)(X,\bar{Y},Z,T) + L((\nabla F)(\bar{X},T),\bar{Y},Z) - a^r L(X,(\nabla F)(Y,T),Z)$$
$$+ a^r P_1(T)L(X,\bar{Y},Z) +$$
$$\frac{n}{(n-2)}\{-a^r(\nabla K)(X,\bar{Y},Z,T) + K((\nabla F)(\bar{X},T),\bar{Y},Z)$$
$$-a^r K(X,(\nabla F)(Y,T),Z) + a^r P_1(T)K(X,\bar{Y},Z) + a^r(\nabla V)(X,\bar{Y},Z,T)$$
$$-V((\nabla F)(\bar{X},T),\bar{Y},Z)\} + a^r V(X,(\nabla F)(Y,T),Z) - a^r P_1(T)V(X,\bar{Y},Z)\} \quad (2.30)$$

If a (1,2)-recurrent H-HSU-Structure manifold is conformal-(1,2) recurrent and conharmonic-(1,2)-recurrent for the same recurrence parameter then from equation (2.30), we get

$$a^r(\nabla V)(X,\bar{Y},Z,T) - V\big((\nabla F)(\bar{X},T),\bar{Y},Z\big) + a^r V(X,(\nabla F)(Y,T),Z) = a^r P_1(T)V(X,\bar{Y},Z).$$

Which shows that the manifolds is concircular –(1,2)-recurrent.
Similarly, it can be shown that if the recurrent manifold is either conformal-(1,2)-recurrent and concircular-(1,2)-recurrent or conharmonic-(1,2)-recurrent and concircular-(1,2)-recurrent then it is either conharmonic-(1,2)-recurrent or conformal-(1,2)-recurrent for same recurrence parameter.

Theorem (2.8)) In a (1,2,3) recurrent H-HSU-structure manifold, if any two of following conditions hold for the same recurrence parameter, then the third also hold:

(b) It is conformal (1,2,3)-recurrent ,
(b) It is conharmonic (1,2,3)-recurrent,
(c) It is concircular (1,2,3)-recurrent.

Barring X,Y and Z in equations (2.22) ,we get

$$C(\bar{X},\bar{Y},\bar{Z}) = L(\bar{X},\bar{Y},\bar{Z}) + \frac{n}{(n-2)}[K(\bar{X},\bar{Y},\bar{Z}) - V(\bar{X},\bar{Y},\bar{Z})]. \quad (2.31)$$

Now, from equation (1.1) and (2.31),we have

$$a^r P_1(T)C(X,\bar{Y},Z) = a^r P_1(T)L(X,\bar{Y},Z) + \frac{na^r}{(n-2)} P_1(T)\{K(X,\bar{Y},Z) - V(X,\bar{Y},Z)\} \quad (2.32)$$

Differentiating equation(2.31) with respect to T, using equation (2.31) and then barring X in the resulting equation, we get

$$-a^r(\nabla C)(X,\bar{Y},\bar{Z},T) + C\big((\nabla F)(\bar{X},T),\bar{Y},\bar{Z}\big) - a^r C(X,(\nabla F)(Y,T),\bar{Z}) - a^r C(X,\bar{Y},(\nabla F)(Z,T)$$
$$= a^r((\nabla L)(X,\bar{Y},\bar{Z},T) \quad + L\big((\nabla \mathrm{F})(\bar{\mathrm{X}},\mathrm{T}),\bar{\mathrm{Y}},\bar{\mathrm{Z}}\big) - a^r L(X,(\nabla F)(Y,T),\bar{Z})$$
$$-a^r L(X,\bar{Y},(\nabla F)(Z,T)) + \frac{\mathrm{n}}{(\mathrm{n}-1)}\{-\mathrm{a}^\mathrm{r}(\nabla\mathrm{K})(\mathrm{X},\bar{\mathrm{Y}},\bar{\mathrm{Z}},\mathrm{T}) + K((\nabla F)(\bar{X},\mathrm{T}),\bar{Y},\bar{Z})$$
$$-a^r K(X,(\nabla F)(Y,T),\bar{Z}) - \big(X,\bar{Y}(\nabla F)(Z,T) + a^r(\nabla V)(X,\bar{Y},\bar{Z},T) - V\big((\nabla F)(\bar{X},T),\bar{Y},\bar{Z}\big)\}$$
$$+a^r V(X,(\nabla F)(Y,T),\bar{Z}) + a^r V(X,\bar{Y}(\nabla F)(Z,T))\}. \quad (2.33)$$

Adding equations (2.32) and (2.33),we get

$$-a^r(\nabla C)(X,\bar{Y},\bar{Z},T) + C\big((\nabla F)(\bar{X},T),\bar{Y},\bar{Z}\big) - a^r C(X,(\nabla F)(Y,T),\bar{Z})$$
$$-a^r C\big(X,\bar{Y}(\nabla F)(Z,T)\big) + a^r P_1(T)C(X,\bar{Y},\bar{Z})$$
$$= -a^r((\nabla L)(X,\bar{Y},\bar{Z},T) \ + L((\nabla \mathrm{F})(\bar{\mathrm{X}},\mathrm{T}),\bar{\mathrm{Y}},\bar{\mathrm{Z}}) - a^r L(X,(\nabla F)(Y,T),\bar{Z})$$
$$-a^r L(X,\bar{Y}(\nabla F)(Z,T)) \qquad + a^r P_1(T)L(X,\bar{Y},\bar{Z})$$
$$+\frac{\mathrm{n}}{(\mathrm{n}-1)}\{-\mathrm{a}^\mathrm{r}(\nabla\mathrm{K})(\mathrm{X},\bar{\mathrm{Y}},\bar{\mathrm{Z}},\mathrm{T})$$
$$+K((\nabla F)(\bar{X},\mathrm{T}),\bar{Y},\bar{Z}) - a^r K(X,(\nabla F)(Y,T),\bar{Z}) - a^r K\big(X,\bar{Y}(\nabla F)(Z,T)\big)$$
$$+a^r P_1(T)K(X,\bar{Y},\bar{Z}) + a^r(\nabla V)(X,\bar{Y},\bar{Z},T) - V((\nabla F)(\bar{X},T),\bar{Y},\bar{Z})\}$$
$$+a^r V(X,(\nabla F)(Y,T),\bar{Z}) - a^r P_1(T)V(X,\bar{Y},\bar{Z})\} \quad (2.34)$$

If a (1,2,3)-recurrent H-HSU-Structure manifold is conformal-(1,2,3) recurrent and conharmonic-(1,2,3)-recurrent for the same recurrence parameter then from equation (2.34), we get

$$a^r(\nabla V)(X,\bar{Y},\bar{Z},T) - V\big((\nabla F)(\bar{X},T),\bar{Y},\bar{Z}\big) + a^r V(X,(\nabla F)(Y,T),\bar{Z}) + a^r V\big(X,\bar{Y}(\nabla F)(Z,T)\big)$$
$$= a^r P_1(T)V(X,\bar{Y},\bar{Z}).$$

Which shows that the manifolds is concircular –(1,2,3)-recurrent.

Similarly, it can be shown that if the recurrent manifold is either conformal-(1,2,3)-recurrent and concircular-(1,2,3)-recurrent or conharmonic-(1,2,3)-recurrent and concircular-(1,2,3)-recurrent then it is either conharmonic-(1,2,3)-recurrent or conformal-(1,2,3)-recurrent for same recurrence parameter.

References

1. C.J. Hsu, "On some structure which are similar to quaternion structure", *Tohoku Math. J.*, (12), 1960, pp. 403-428.
2. Q. Khan, "On recurrent Riemannian manifolds", *Kuungpook Math. J.*, (44), 2004, pp. 269-276.

3. R.S. Mishra, S.B. Pandey and U. Sharma "Pseudo n-Recurrent Manifold with KH-Structure", *Indian J. pure applied Mathematics*, 7(10), 1974, pp. 1277-1287.
4. R.S. Mishra, "A course in tensors with applications to Riemannian Geometry", II edition Pothishala Pvt., Ltd. Allahabad (1973).
5. R.S. Mishra, "Structure on a differentiable manifold and their applications", Chandrama Prakashan, 50-A, Balrampur House, Allahabad (1984).
6. S.B. Pandey and Lata Dasila," On General Differentiable Manifold". *U. Scientist Phyl. Sciences*, Vol.7, No. 2,147-152(1995).
7. S.B. Pandey and Lata Dasila, "On Birecurrent General Differentiable Manifold". *U. Scientist Phyl. Sciences*, Vol.9, No.1, 1997.