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EXTENSION AND GENERALISATION OF AN INEQUALITY OF PAUL TURAN'S THEOREM ON POLYNOMIAL

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ABSTRACT. In this paper, certain new results concerning the maximum modulus of the polar derivative of a polynomial with restricted zeros are obtained. These estimates strengthen some well known inequalities for polynomial due to Turån, Dubinin and others.

1. INTRODUCTION

In scientific disciplines like physics, engineering, computer science, biology, physical chemistry, economics, and other applied areas, experimental observations and investigations when translated into mathematical language are called mathematical models. The solution of these models could lead to problems of estimating how large or small the maximum modulus of the derivative of an algebraic polynomial can be in terms of the maximum modulus of that polynomial. Bounds for such type of problems are of some practical importance. Since, there are no closed formulae for precise evaluation of these bounds and whatever is available in literature is in the form of approximations. However for practical purposes, nobody ever needs exacts bounds and mathematicians must only indicate methods for obtaining approximate bounds. These approximate bounds, when computed efficiently, are quite satisfactory for the needs of investigators and scientists. Therefore there is always a desire to look for better and improved bounds than those available in literature. It is this aspiration of obtaining more refined and revamped bounds that has inspired our work in this article. In this paper, we have generalized and refined some well known results concerning the polynomials due to Turån [16], Dubinin [6] and others. Let begin with the polynomial of the form $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree n > 1

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and let P' (z) be the derivative of P (z). Then concerning the lower bound for the maximum of P'(z)l in terms of maximum of for class of polynomials P G {Pn not vanishing outside unit disc, Turån [16] showed that

(1.1) $$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|.$$

Equality in inequality (1.1) holds for those polynomials P G {Pn which have all their zeros on Izl = 1. As an extension of (1.1), Govil [8] proved that if P G (Pn and P (z) has all its zeros in Izl k, k 1, $k \geq 1$, then

(1.2) $$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|.$$

In literature, there exist several generalizations and extensions of (1.1) and (1.2) (see [1, 2, 4, 5, 13—15]). Dubinin [6] refined inequality (1.1) by proving that if all the zeros of P e {Pn lie in 1, then

(1.3) $$\max_{|z|=1} |P'(z)| \geq \frac{1}{2}\left(n + \frac{\sqrt{|a_n|} - \sqrt{|a_0|}}{\sqrt{|a_n|}}\right) \max_{|z|=1} |P(z)|.$$

The polar derivative DOP(z) of P e {Pn with respect to the point a G C is defined by

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

The polynomial DnP(z) is of degree at most n — I and it generalizes the ordinary derivative P'(z) of P (z) in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z),$$

uniformly for Izl R, R > O.

A. Aziz [Il, Aziz and Rather ([4, 5]) obtained several sharp estimates for maximum modulus of DuP(z) on Izl = I and among other things they extended inequality (1.2) to the polar derivative of a polynomial by showing that if P e (Pn has all its zeros in Izl k, k 1, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$

(1.4) $$\max_{|z|=1} |D_\alpha P(z)| \geq n\left(\frac{|\alpha| - k}{1+k^n}\right) \max_{|z|=1} |P(z)|.$$

2. MAIN RESULTS

In this paper, we obtain certain refinements and generalizations of inequalities (1.1), (1.2), (1.3) and (1.4). We first prove the following result.

Theorem 2.1. If P e [Pn and P (z) has all its zeros in 121 k, k 1, then for every α G C, with k

$$|D_\alpha P(z)| \geq \frac{|\alpha|-k}{1+k^n}\left(n+\frac{\sqrt{k^n|a_n|}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|}}\right)\left(\max_{|z|=1}|P(z)|+\frac{|a_{n-1}|\phi(k)}{k}\right) + \psi(k)|na_0+\alpha a_1|,$$

max (2.1) 21=1 where +(k) $=\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right)$ or (k-1)2 and q/'(k) 1 or I— according as n > 2 or n = 2.

Remark 2.1. Since all the zeros of p (z) lie in zl k, k 2 1, it follows that laol ≤k ^{n} an l.

In view of this, inequalities (2.1) refines inequality (1.4).

If we divide the two sides of inequality (2.1) by la and let al 00, we get the following result.

1, then

(2.2) $$\max_{|z|=1}|P'(z)| \geq \frac{1}{1+k^n}\left(n+\frac{\sqrt{k^n|a_n|}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|}}\right)\left(\max_{|z|=1}|P(z)|+\frac{|a_{n-1}|}{k}\phi(k)\right) + \psi(k)|a_1|,$$

where $=\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right)$ or $\frac{(k-1)^2}{2}$ and $\psi(k)=1-\frac{1}{k^2}$ or $1-\frac{1}{k}$ according as n > 2

Corollary 2.1. If P 'J)n and P (z) has all its zeros in Izl k, k

or n = 2.

The result is best possible and equality in inequality (2.2) holds for P (z) = $z^n + k^n$.

Remark 2.2. As before, it can be easily seen that inequality (2.2) refines inequality (1.2). Further for k: = I, inequality (2.2) reduces to inequality (1.3).

Next, we present the following result which is generalisation of Theorem 2.1 and in particular, includes refinement of inequality (1.2) as a special case.

Theorem 2.2. If all the zeros of P e 'J)n lie in Iz k, k 2 1, then for every a C with al 2 k, Of I < 1,

$$\max_{|z|=1}|D_\alpha P(z)| \geq \frac{n}{1+k^n}\left\{(|\alpha|-k)\max_{|z|=1}|P(z)|+(|\alpha|+1/k^{n-1})lm\right\}$$

(2.3)

$$+\frac{(|\alpha|-k)}{k^n(k^n+1)}\left(\frac{\sqrt{k^n|a_n|-lm}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|-lm}}\right)\left(k^n\max_{|z|=1}|P(z)|-lm\right.$$

$$+\frac{(|\alpha|-k)|a_{n-1}|\phi(k)}{k(1+k^n)}\left(n+\frac{\sqrt{k^n|a_n|-lm}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|-lm}}\right)$$

$$+\psi(k)|na_0+\alpha a_1|,$$

where $m = \min_{|z|=k}|P(z)|,\ \phi(k) = \left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right)$ or $\frac{(k-1)^2}{2}$ and $\psi(k) = 1 -$ or

1 — according as n > 2 or n = 2.

If we divide both sides of inequality (2.3) by al and let → 00, we get the following result.

Corollary 2.2. If all the *l the zeros of* $P \in \mathcal{P}_n$ *lie in* $|z| \leq k,\ k \geq 1$*, then f* zeros of P 1, then for O I < I

max P (z)1

Izl=l

$$\geq \frac{n}{1+k^n}\left\{\max_{|z|=1}|P(z)|+lm\right\}+\psi(k)|a_1|$$

(2.4) — 1m

$$+\frac{1}{k^n(k^n+1)}\left(\frac{\sqrt{k^n|a_n|-lm}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|-lm}}\right)\left(k^n\max_{|z|=1}|P(z\right.$$

$$+\frac{|a_{n-1}|}{k(1+k^n)}\left(n+\frac{\sqrt{k^n|a_n|-lm}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|-lm}}\right)\phi(k),$$

$P(z)|,\ \phi(k) = \left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right)$ or $\frac{(k-1)^2}{2}$ and q/(k) = I

where m = minlzl=k — or I — according as n > 2 or n = 2.

Remark 2.3. For I = O, Corollary 2.2 reduces to Corollary 2.1 and for k = I, inequality (2.4) refines inequality (1.3).

3. LEMMAS

We need the following lemmas for the proof of our theorems. The first lemma is due to Dubinin[6].

Lemma 3.1. If P (Pn and P (z) has all its zeros in zl I, then

(3.1) $$|P'(z)| \geq \frac{1}{2}\left(n + \frac{\sqrt{|a_n|} - \sqrt{|a_0|}}{\sqrt{|a_n|}}\right)|P(z)|,$$ for Izl = I.

The next lemma is special case of a result due to Aziz and Rather [3, 4]. Lemma 3.2. If P (Pn and P (z) has its all zeros in zl I, then for zl = I

$$|Q'(z)| \leq |P'(z)|,$$

where Q(z) = z^n p (I/Z).

Lemma 3.3. If all the zeros of P G {Pn lie in a circular region C and w is any zero of DuP(z), the polar derivative of P(z), then at most one of the points w and a may lie outside C.

The above lemma is due to Laguerre (see [10]). The following lemma is due to

Frappier, Rahman and Ruscheweyh [7].

Lemma 3.4. If P(z) is a polynomial of degree at most n

(3.2) max - (IV $|P(z)| \leq R^n \max_{|z|=1} |P(z)|$ -

Izl=R

and

(3.3) max _ R max IP(z) $| - (R-1)|P(0)|,$

Izl=R

Next lemma is the famous result of P. D. Lax [9].

1, then for R 2 1 if n 2 2,

if n = 1.

Lemma 3.5. If P G {Pn does not vanish in Izl < 1, then

$'(z)| \leq$ — max Il) (z) l, for Izl= 1.

We also need the following lemma.

Lemma 3.6. *If $P(z) = a_n \prod_{j=1}^{n}(z - z_j)$ is a polynomial of degree* n If P (z) an 2 having *, then for $R \geq 1$* no zero in Izl < 1, then for R

$$|P(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |P(z)| - \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2}\right)|P'(0)|,$$

(3.4) maxif n > 2,
Izl=R

and

(3.5) $\max_{|z|=R} |P(z)| \leq \frac{R^2+1}{2} \max_{|z|=1} |P(z)| - \frac{(R-1)^2}{2}|P'(0)|,$

Proof of Lemma 3.6. For each 0, O O < Yr, we have

(3.6) $P(Re^{i\theta}) - P(e^{i\theta}) = \int_1^R e^{i\theta} P'(te^{i\theta})dt,$

which gives with the help of (3.2) of Lernrna 3.4 and Lernrna 3.5 for n > 2

$$|P(Re^{i\theta}) - P(e^{i\theta})| \leq \int_1^R |P'(te^{i\theta})| dt$$

$$\leq \frac{n}{2} \int_1^R t^{n-1} dt \max_{|z|=1} |P(z)| - \int_1^R (t^{n-1} - t^{n-3})dt|P'(0)|$$

$$= \frac{R^n - 1}{2} \max_{|z|=1} |P(z)| - \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2}\right)|P'(0)|.$$

Consequently for n > 2 and O O < 27T, we have

$$|P(Re^{i\theta})| \leq |P(Re^{i\theta}) - P(e^{i\theta})| + |P(e^{i\theta})|$$

$$< \frac{R^n + 1}{2} \max_{|z|=1} |P(z)| - \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2}\right)|P'(0)|,$$

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_ 2 121=1 which immediately leads to (3.4). Similarly we can prove inequality (3.5) by using inequality (3.3) of Lemma 3.4. This proves Lemma 3.6.

Finally we require the following lemma.

Lemma 3.7. If P 1, has all its zeros in Izl $\le k$, where $k \ge 1$, then for 0 I < I

max > $\frac{2k^n}{1+k^n}\max_{|z|=1}|P(z)| + l\left(\frac{k^n-1}{k^n+1}\right)\min_{|z|=k}|P(z)|$

212k

(3.7) if n > 2, $+\frac{2k^{n-1}|a_{n-1}|}{k^n+1}\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right)$, *if*

and

(3.8)

'2k2 k2 _ 1— 1) 2 1a11 maxmax + 1 min +if n 2.

Izl=k• 1 + k'2 121—1 k.2 + 1

Proof of Lemma 3.7. Since all the zeros of P G [Pn lie in Izl k, k 1, therefore, all the zeros of g(z) P(kz) lie in Izl 1 and hence all the zeros of f (z) z^nTFj = z^n P(k/Z) lie in Izl 1. Moreover, m = milllzl=k $|P(z)|$ =milllzl—l so that rnlznl for Izl = 1.

We show that for G C with < l, f (z) 4- /\m$z^n \neq$ 0 in Izl < 1. This is trivially true if m 0. Henceforth we suppose that m 0, so that all the zeros of f (z) lie in Izl > 1. By the maximum modulus theorem

(3.9) mlz^n l < for Izl < 1.

Now if there is point z zo with Izol < 1, such that f (zo) + $\lambda m z_0^n$= 0, then $|f(z_0)|$

= l/\l lzålrn < Izålm,

a contradiction to inequality (3.9). Hence, it follows that the polynomial T (z) = f (z) -F does not vanish in Izl < 1. Applying inequality (3.4) of Lemrna 3.6 to the polynornial T (z), with $R = k \ge 1$ R k 1 and ve get for $|z| = 1$, n > 2, we get for Izl I,

If(kz) + ^{n}mz ^{n} I+ $\le\frac{k^n+1}{2}|f(z)$ /\m$z^n$ l — $\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right)|f'(0)|$.

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Which implies, for n > 2,

(3.10) If(kz) + $\lambda m k^n z^n| \leq \frac{k^n+1}{2}\left(|f(z)|+|\lambda m|\right)-\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right)|f'(0)|.$

Choosing argument of suitably in the left hand side of inequality (3.10), we get for

$|f(kz)|+|\lambda|mk^n \leq \frac{k^n+1}{2}\left(|f(z)|\right.$ + 18m) - ($\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\Big)k^{n-1}|$ an—ll.

Replacing $f(z)$ by $z^n\overline{P(k/\overline{z})}$, we obt we obtain for n > 2 and Izl I

k^nmax Izl=l + I Almk•n $\leq \frac{k^n+1}{2}\left(\max_{|z|=k}|P(z)|+|\lambda|m\right)$
$-\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right)k^{n-}$ I lan—I|,

which on simplification yields inequality (3.7). In a similar manner we can prove inequality (3.8) by applying inequality (3.5) of Lemma 3.6 instead of inequality (3.4) to the polynomial T (z). This proves Lemma 3.7.

4. Proof of the Theorems

Proof of Theorem 2.1. Lot f (z) = P(kz). Since P e [Pn has all its zeros in Izl k whorc k 1, thcrcforc, f e (Pn and f (z) has all its zeros in Izl 1. If Q(z) = z^n f(l/z), thon it is oasy to verify that

(4.1) $|Q'(z)| = |nf(z) - zf'(z)|,$ for Izl = 1.

Combining (4.1) with Lemma 3.2, wc got

(4.2) $|f'(z)| \geq |nf(z) - zf'(z)|,$ for Izl = 1.

Now for cvcry a G C with al 2 k, wc havc for zl = 1 $f(z)| =$ |nf(z) + (a/k —
2 a/kl f'(z)l — In f (z) $- zf'(z$

which gives with the help of (4.2)

(4.3) $|D_{\alpha/k}f(z)| \geq \left(\frac{|\alpha|-k}{k}\right)|f'(z)|.$

Consequently,

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(4.4) $\max_{|z|=k} |D_\alpha P(z)|$ 2 (al
— k) max P
(z)l ׀zl=k .

Again since all the zeros of f (z) = P(kz) lic in Izl 1, thcrcforc, using Lemma 3.1, wc have

$$|f'(z)| \geq \frac{1}{2}\left(n + \frac{\sqrt{k^n|a_n|} - \sqrt{|a_0|}}{\sqrt{k^n|a_n|}}\right)|f(z)|,$$ for zl = I.

Replacing f (z) by P(kz), wc obtain

$$k \max_{|z|=1} |P'(kz)| \geq \frac{1}{2}\left(n + \frac{\sqrt{k^n|a_n|} - \sqrt{|a_0|}}{\sqrt{k^n|a_n|}}\right)$$ max$P(kz$
zl=l

which implics

(4.5) max 2 $\frac{1}{2k}\left(n + \frac{\sqrt{k^n|a_n|} - \sqrt{|a_0|}}{\sqrt{k^n|a_n|}}\right)$ max$|P(z)|$.
Izl=kIzl=k

Combining incquality (4.4) and inequality (4.5), wc havc

$$\max_{|z|=k} |D_\alpha P(z)| \geq \frac{(|\alpha| - k)}{2k}\left(n + \frac{\sqrt{k^n|a_n|} - \sqrt{|a_0|}}{\sqrt{k^n|a_n|}}\right)\max_{|z|=k} |P(z)|.$$
(4.6)

Further sincc Do P (z) is a polynomial of dcgrcc at most n — I, using inequality (3.2) of Lemma 3.4, wc havc for n > 2

$$\max_{|z|=R} |D_\alpha P(z)| \leq R^{n-1} \max_{|z|=1} |D_\alpha P(z)|$$ - (IV 1 - RTE 3)Inao .

Using this inequality and inequality (3.7) of Lemma 3.7 with I — 0 and R k > I in (4.6), we have for n > 2

$$k^{n-1}\max_{|z|=1}|D_\alpha P(z)|-(k^{n-1}-k^{n-3})|na_0+\alpha a_1|$$
$$\geq\frac{(|\alpha|-k)}{2k}\left(n+\frac{\sqrt{k^n|a_n|}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|}}\right)$$
$$\times\left\{\frac{2k^n}{1+k^n}\max_{|z|=1}|P(z)|+\frac{2k^{n-1}|a_{n-1}|}{k^n+1}\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right)\right\},$$

which on simplification gives

$$|D_\alpha P(z)|\geq\frac{|\alpha|-k}{1+k^n}\left(n+\frac{\sqrt{k^n|a_n|}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|}}\right)$$
$$\times\left\{\max_{|z|=1}|P(z)|+\frac{|a_{n-1}|}{k}\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right)\right\}$$
$$+\left(1-1/k^2\right)|na_0+\alpha a_1|,\quad \text{if } n>2.$$

max
121=1

The above inequality is equivalent to the inequality (2.1) for n > 2. For n = 2, the result follows on similar lines by using inequality (3.3) of Lemma 3.4 and inequality (3.8) of Lemma 3.7 in the inequality (4.6). This cornpletes the proof of Theorem 2.1.

Proof of Theorem 2.2. By hypothesis P e has all zeros in Izl k, k 2 1. If P (z) has a zero on Izl = k, then m = minlzl=k = 0 and result follows from Theorem 2.1. Henceforth, we suppose that P (z) has all its zeros in Izl < k, k 2 1, so that m > O. Now if f (z) = P(kz), then f e {Pn and f (z) has all zeros in zl < I and m = minlzl=k = milllzl=l This implies

$f(z)|$,for Izl= 1.

By the Rouche's Theorem, we conclude that for every e C with < I, the polynomial g(z) = f (z) — has all zeros in Izl < I. Applying inequality (4.3) to the polynomial g(z), it follows for Izl = I and lal

Since all the zeros of g(z) lie in Izl < I, using Lemma 3.1, we obtain for Izl = I and lal 2k

$$|D_{\alpha/k}g(z)| \geq \frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(n+\frac{\sqrt{|k^n a_n-\lambda m|}-\sqrt{|a_0|}}{\sqrt{|k^n a_n-\lambda m|}}\right)|g(z)|.$$

Using the fact that the function S(x) x > 0, is non-decreasing function of x and lc^n an — 2 Ic^n lanl we get for every e C with < I and z 1

$$|D_{\alpha/k}g(z)| \geq \frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(n+\frac{\sqrt{k^n|a_n|-|\lambda|m}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|-|\lambda|m}}\right)|g(z)|.$$

(4.7)

Replacing g(z) by f (z) — in (4.7), we get for Izl = I and al 2k

$$-\frac{nm\alpha\lambda}{k}z^{n-1}\Big|$$

Do/kf(z) -

(4.8)

1 101 $$\Big)\left(n+\frac{\sqrt{k^n|a_n|-|\lambda|m}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|-|\lambda|m}}\right.$$ — k) (If(z) - Aml).

Since all the zeros of f (z) — /\mz^n = g(z) lie in Izl < 1 and a/ kl 1, it follows by Lemma 3.3 that all the zeros of

nmav\

$_{/k}(f(z)$- mxzn) = Da/kf(Z) - z $_{n-}$k lie in

Iz < 1. This irnplies that

(4.9)for zl > 1.

In view of this inequality, choosing argument of in the left hand side of inequality (4.8) such that

$$\left|D_{\alpha/k}f(z) - \frac{nm\alpha\lambda}{k}z^{n-1}\right| = \left|D_{\alpha/k}f(z)\right| - \frac{nm|\alpha||\lambda|}{k}, \quad \text{for } |z| = 1,$$

we get for $|z| = 1$ and $|\alpha| \geq k$

$$\left|D_{\alpha/k}f(z)\right| - \frac{nm|\alpha||\lambda|}{k}$$
$$\geq \frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(n + \frac{\sqrt{k^n|a_n| - |\lambda|m} - \sqrt{|a_0|}}{\sqrt{k^n|a_n| - |\lambda|m}}\right)\left(|f(z)| - |\lambda|m\right),$$

which on simplification yields

$$\left|D_{\alpha/k}f(z)\right| \geq \frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(n + \frac{\sqrt{k^n|a_n| - |\lambda|m} - \sqrt{|a_0|}}{\sqrt{k^n|a_n| - |\lambda|m}}\right)|f(z)|$$
$$- \frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(\frac{\sqrt{k^n|a_n| - |\lambda|m} - \sqrt{|a_0|}}{\sqrt{k^n|a_n| - |\lambda|m}}\right)|\lambda|m + \frac{n}{2}\left(\frac{|\alpha|+k}{k}\right)|\lambda|m.$$

This implies for Izl — I and [al 2 k

(4.10)

$$|D_\alpha P(z)| \geq \frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(n + \frac{\sqrt{k^n|a_n| - |\lambda|m} - \sqrt{|a_0|}}{\sqrt{k^n|a_n| - |\lambda|m}}\right) \text{ max}$$
$$- \frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(\frac{\sqrt{k^n|a_n| - |\lambda|m} - \sqrt{|a_0|}}{\sqrt{k^n|a_n| - |\lambda|m}}\right)|$$

max zIzl=kIzl=k

I Alm +

$$\frac{n}{2}\left(\frac{|\alpha|+k}{k}\right)|\lambda|m.$$

Moreover, since DaP(z) is a polynomial of degree at most n — 1, applying inequality (3.2) of Lemma 3.4 and inequality (3.7) of Lemma 3.7 with R = k 2 1, we obtain for lal 2 k, 0 s I < 1 and Izl = 1

$$
\begin{aligned}
& k^{n-1}\max_{|z|=1}|D_\alpha P(z)| - (k^{n-1}-k^{n-3})|na_0+\alpha a_1| \\
\geq & \frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(n+\frac{\sqrt{k^n|a_n|-lm}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|-lm}}\right) \\
& \times\left\{\frac{2k^n}{1+k^n}\max_{|z|=1}|P(z)|+\frac{k^n-1}{k^n+1}lm+\frac{2k^{n-1}|a_{n-1}|}{k^n+1}\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right)\right\} \\
& -\frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(\frac{\sqrt{k^n|a_n|-lm}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|-lm}}\right)lm+\frac{n}{2}\left(\frac{|\alpha|+k}{k}\right)lm, \quad \text{if } n>2.
\end{aligned}
$$

Equivalently, we have for $|\alpha|\geq k,\ 0\leq l<1$ and $|z|=1$

$$
\begin{aligned}
\max_{|z|=1}|D_\alpha P(z)| \geq & \frac{n}{1+k^n}\left\{(|\alpha|-k)\max_{|z|=1}|P(z)|+(|\alpha|+1/k^{n-1})lm\right\} \\
& +\frac{(|\alpha|-k)}{k^n(k^n+1)}\left(\frac{\sqrt{k^n|a_n|-lm}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|-lm}}\right)\left(k^n\max_{|z|=1}|P(z)|-lm\right. \\
& +\frac{(|\alpha|-k)|a_{n-1}|}{k(1+k^n)}\left(n+\frac{\sqrt{k^n|a_n|-lm}-\sqrt{|a_0|}}{\sqrt{k^n|a_n|-lm}}\right) \\
& \times\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right)+(1-1/k^2)|na_0+\alpha a_1|, \quad \text{if } n>2.
\end{aligned}
$$

That proves the inequality (2.3) for n > 2. For the case n = 2, the result follows on similar lines by using inequality (3.3) of Lemma 3.4 and inequality (3.8) of Lemma 3.7 in the inequality (4.10). This completes the proof of Theorem 2.2.

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