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Local Convergence of Two Fifth Order Algorithms with H$\ddot{o}$Lder Continuity Assumptions

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Abstract
In order to estimate the solution of the zero for the nonlinear systems, we conduct the local convergence investigation in this paper. In contrast to the Lipschitz condition used in the preceding study, we have used the H$\ddot{o}$lder continuity requirement. Additionally, we use a derivative approximation to take the derivative free iterative technique with the same order. A computed radius of convergence balls based on the H$\ddot{o}$lder constant is also provided. No Taylor's series approximation on a higher order Fr$\acute{e}$chet derivate is used in this investigation. To broaden the relevance of our work, a comparison of convergence ball radii is also provided. This highlights the uniqueness of this paper.

Keywords: Nonlinear equations, iterative methods, local convergence, divided differences.

AMS Subject Classification: 65H05, 65H10.

1. Introduction

In this paper, we concerned with the problem of approximating a locally unique solution α of the nonlinear system

$$G(x) = 0. \tag{1}$$

Solving nonlinear system of equations play an important role in many branches of nonlinear Functional analysis, Numerical Analysis, Chemical engineering, Kinetic theory of gases [1, 2, 3, 4, 5, 6], etc. Many nonlinear problems arise from discretization of nonlinear integral equations and nonlinear differential equations by method of finite difference. In the literature, we can find several real world problems described by nonlinear models which can be transformed into system of nonlinear equations. Such nonlinear model are like variational inequalities, Bratu's problem, a shallow arch, etc. find in the paper [7]. However, most of the equations are phrased in terms of system of nonlinear equations of form (1).

The nonlinear system's relevance to the issue of analysing the coarse-grained dynamical characteristics of neural networks in kinetic theory is covered in [8]. Additionally, Nejat and Ollivier [9] raised the issue to investigate the impact of discretization order on high-order Newton-Krylov unstructured flow solver preconditioning and convergence. In [10], Grosan and Abraham demonstrated how the system of nonlinear equations may be used to solve problems in neurophysiology, kinematic syntheses, chemical equilibrium, combustion, and economic modelling. The reactor and steering problems were just recently solved by Awawdeh and Tsoulos et al. [11, 12] by rewriting them as systems of nonlinear equations.

This type of nonlinear equations can be approximatively solved using a variety of traditional approaches. The main justification for developing iterative approaches is that most types of nonlinear equations frequently lack an analytical solution. These iterative methods, which can be split into single-point, multi-point, with memory, and without memory methods, are currently being explored. As a matter of fact, numerous higher order multipoint iterative techniques for solving nonlinear equations have been developed and published in a number of publications of applied and computer mathematics. In order to increase the order of convergence, most new methods start by using the well-known quadratically convergent Newton's method to solve nonlinear equation (1). Numerous authors have created iterative techniques that are reliable and effective with higher convergence orders, however it is crucial to talk about local and semi-local convergence analysis for them.

The study about local convergence of higher order iterative methods can be analyzed under different continuity conditions in Banach spaces (see, [13, 14]). Argyros and George [15] developed the local convergence analysis of third order Halley-like methods under Lipchitz continuity conditions and it is given for k = 0, 1, 2, . . . by

$$\begin{aligned} y_k &= x_k - G'(x_k)^{-1}G(x_k) \\ u_k &= y_k + (1-a)G'(x_k)^{-1}G(x_k) \\ z_k &= y_k - \gamma A_{a,k}G'(x_k)^{-1}G(x_k) \\ x_{k+1} &= z_k - \eta B_{a,k}G'(x_k)^{-1}G(z_{k)}, \end{aligned}$$

Where,

$\eta, \gamma, \alpha \, \epsilon(-\infty,\infty) - \{0\}, H_{a,k} = \frac{1}{a}G'(x_k)^{-1}\big(G'(u_k) - G'(x_k)\big), A_{a,k} = I - \frac{1}{2}H_{a,k}\left(I - \frac{1}{2}H_{a,k}\right), B_{a,k} = I - H_{1,K} + H^2{}_{a,k}$.

The local convergence of Chebyshev-Halley-type method discussed in [17] and in is given for $k = 0,1,2 \ldots. by$

$$\begin{aligned} y_k &= x_k - G'(x_k)^{-1}G(x_k) \\ z_k &= x_k - \big(1 + (G(x_k) - 2\eta G(y_k))^{-1}G(y_k)\big)G'(x_k)^{-1}G(x_k) \\ x_{k+1} &= z_k - \left(G'(x_k) + \bar{G}''(x_k)(z_k - x_k)\right)^{-1} G(z_k), k \geq 0, \end{aligned}$$

Where, $\bar{G}''(x_k) = 2(y_k)G'(x_k)^2G(x_k)^{-2}$ and η is a parameter. The order of this family is at least five for any value of η and for $\eta = 1$, it is six.

In this paper, we analyze the local convergence of fifth order iterative method which is studied in [20] under the condition that first order Fréchet derivative satisfying the Lipschitz continuity condition. We have used the weaker convergence condition for this purpose. We have utilized the Hölder continuity condition in place of Lipschitz condition. The existence and uniqueness region of the solution is also established. Numerical examples worked out and convergence balls for each of them are obtained. We compare these results with the convergence balls of existing methods (2) and (3). Also, we discuss the local convergence of derivative free iterative method obtained by approximating the derivative by divided differences. Some numerical examples worked out and the convergence regions computed.

This paper is divided into four sections and organized as follows. In Section 1, we form the introduction. The local convergence study is performed in Section 2. The existence and uniqueness region of the solution is derived along with some numerical examples. In Section 3, the local convergence of the derivative free iterative method is discussed and also the computation of existence and uniqueness region of solution with numerical examples. Finally, the conclusion forms Section 4.

2. Local convergence analysis

In this section, we consider a fifth order iterative method proposed in [16] and its local convergence analysis under Lipschitz conditions on G'. It is given for $k = 0,1,2,\dots$ by

$$y_k = x_k - G'(x_k)^{-1}G(x_k)$$
$$z_k = x_k - 2(G'(x_k) + G'(y_x))^{-1}G(x_k)$$
$$z_{k+1} = z_k - G'(y_k)^{-1}G(z_k),$$

where x_0 is the starting point. In [16], Cordero et al. presented the fifth order of convergence using Taylor series on higher order Frechet derivative without obtaining the convergence balls. They also assumed that ´ the starting point x0 is sufficiently close to the solution without estimating this closeness. Now, we have addressed these problems using only first order Fr$\acute{e}$chet derivative.

Suppose that $B(\upsilon,\rho)$ and $\bar{B}(\upsilon,\rho)$ denote the open and closed balls, respectively with center υ and radius ρ Let $F : D \subseteq X \to Y$ be a Frechet differentiable operation defined on open domain D such that for $\alpha \in D, L_0 > 0, L > 0$ and for all $x, y \in D$, we have

$$G(\alpha) = 0, G'(\alpha)^{-1} \in L(Y,X), G'(\alpha)^{-1} \neq 0 \quad (5)$$
$$\|G'(\alpha)^{-1}(G'(x) - G'(\alpha))\| \leq L_0\|x - \alpha\|, \forall x \in D \quad (6)$$
$$\left\|G'(\alpha)^{-1}(G'(x) - G'(y))\right\| \leq L\|x - y\|, \forall x, y \in B\left(\alpha, \frac{1}{L_0}\right) \subseteq D. \quad (7)$$

Lemma1. *if the nonlinear operator G satisfies the above assumptions, then for all* $x, \in B\left(\alpha, \frac{1}{L_0}\right)$ *we have*

$$\|G'(\alpha)^{-1}(G'(x) - G'(\alpha))\| \leq 1 + L_0\|x - \alpha\|,$$
$$\|G'(\alpha)^{-1}G'(\alpha + t(x - \alpha))\| \leq 1 + L_0\|x - \alpha\|, \forall t]0,1[,$$
$$\|G'(\alpha)^{-1}G'(x)\| \leq (1 + L_0\|x - \alpha\|)\|x - \alpha\|.$$

Proof. The proof is trivial and can be seen in[13].

The following result describes the local convergence theorem for the iterative method (4)

Theorem1. Let G be a nonlinear operator satisfying assumptions (5) , (6)and(7).Then ,the sequence $\{x_{k+1}\}$ generated by (4)is well defined for $x_0 \in B(\alpha, r_3$andconvergesto α ,where,$r_3$isthesmallest positive root of s_3.Also,we obtain the following inequalities for k

$k \geq 0$:

$$\|y_k - \alpha\| \leq g_1(\|x_k - \alpha\|)\|x_k - \alpha\| < \|x_k - \alpha\| < r_{3,} \quad (8)$$
$$\|z_k - \alpha\| \leq g_2(\|x_k - \alpha\|)\|x_k - \alpha\| < \|x_k - \alpha\| < r_{3,} \quad (9)$$
$$\|x_k - \alpha\| \leq g_3(\|x_k - \alpha\|)\|x_k - \alpha\| < \|x_k - \alpha\| < r_{3,} \quad (10)$$

where, s_3, g_1, g_2 and g_3 are auxiliary functions defined in the proof. If there exists a $R \in [r_3, \frac{2}{L_0})$ such that $\overline{B}(\alpha, R) \subseteq D$, then α is the unique solution in $\overline{B}(\alpha, R)$.

Proof. Since $x_0 \in D$ and using (6), we get

$$\|I - G'(\alpha)^{-1}G'(x_0)\| = \|G'(\alpha)^{-1}G'(x_0) - G'(\alpha)\| \leq L_0\|x_0 - \alpha\| < 1$$

For $\|x_0 - \alpha\| < \frac{2}{L_0}$. Therefore, by Banach Lemma, $F'(x_0)^{-1}$ exists and

$$\|G'(x_0)^{-1}G'(\alpha)\| \leq \frac{1}{1 - L_0\|x_0 - \alpha\|}. \quad (11)$$

Thus, y_0 is well defined. From the first equation of (4) for $k = 0$, we get

$$\begin{aligned} y_0 - \alpha &= x_0 - \alpha\, G'(x_0)^{-1}G(x_0) \\ &= -G'(x_0)^{-1}(G(x_0) - G'(x_0)(x_0 - \alpha)) \\ &= -G'(x_0)^{-1}G'(\alpha)\int_0^1 G'(\alpha)^{-1}\left[G'\big(\alpha + t(x_0 - \alpha)\big) - G'(x_0)\right](x_0 - \alpha)dt \end{aligned}$$

By using adequately Banach Lemma, the assumptions and denoting $e_0 = \|x_0 - \alpha\|$, we have

$$\|y_0 - \alpha\| \leq \frac{L\|x_0 - \alpha\|}{2(1 - L_0\|x_0 - \alpha\|)}\|x_0 - \alpha\| \leq g_1(e_0)e_0,$$

(12)

Where,

$$g_1(t) = \frac{Lt}{2(1 - L_0 t}.$$

Obviously g_1 is an increasing function, and by taking $r_1 = \frac{2}{L + 2L_0}$ it follows:

$$0 \leq (t) < 1, \forall t \in [0, r_1).$$

(13)

Using (12) and (13), we get

$$\|y_0 - \alpha\| \leq g_1(\|x_0 - \alpha\|)\|x_0 - \alpha\| < \|x_0 - \alpha\|.$$

Again using (4) for $k = 0, we\ get$

$$z_0 - \alpha \;=\; x_0 - \alpha - 2(G'(y_0) + G'(x_0))^{-1}G(x_0)$$

$$= (G'(y_0) + G'(x_0))^{-1}[(G'(y_0) + G'(x_0)(x_0 - \alpha) - 2G(x_0)]$$

$$= (G'(y_0) + G'(x_0))^{-1}[(G'(x_0) + G'(x_0)(x_0 - \alpha) + G(x_0) - G'(y_0)(x_0 - \alpha)]$$

$$= (G'(y_0) + G'(x_0))^{-1}G'(\alpha)\left[2G'(\alpha)^{-1}\int_0^1 [G'\big(\alpha + t(x_0 - \alpha)\big) - G'\,(x_0)](x_0 - \alpha)dt \right.$$
$$\left. + G'(\alpha)^{-1}(G'(y_0))(x_0 - \alpha)\right]$$

To follow with the study of existence and bound for the product $(G'(y_0) + G'(x_0))^{-1}G'(\alpha)$ we observe that

$$-\frac{1}{2}G'(\alpha)^{-1}[G'(x_0) + G'(y_0) - 2G'(\alpha)] = I - \underbrace{\frac{1}{2}G'(\alpha)^{-1}[G'(x_0) + G'(y_0)]} = I - A\,,$$

So we try to apply Banach lemma

$$\begin{aligned} \|I - A\| &= \left\|\frac{1}{2}G'(\alpha)^{-1}[G'(x_0) + G'(y_0) - 2G'(\alpha)]\right\| \\ &= \left\|\frac{1}{2}[G'(\alpha)^{-1}\big(G'(x_0) - G'(\alpha)\big) + G'(\alpha)^{-1}\big(G'(y_0) - G'(\alpha)\big)]\right\| \\ &\leq \frac{1}{2}(L_0\|x_0 - \alpha\| + L_0\|y_0 - \alpha\|) \\ &\leq \frac{1}{2}(L_{0^{e_0}} + L_{0^{g_1}}(e_0)e_0) = P_1(e_0) \end{aligned}$$

Where $P_1(t) = \frac{1}{2}L_0(1 + g_1(t))t$ is an increasing function such that $P_1(0) = 0$ and $p_1(r_1) = \frac{1}{2}L_{0^{r_1}}(1 + g_1(r_1) = L_{0^{r_1}} < 1$ and so one has:

$$\|2(G'(x_0) + G'(y_0))^{-1}G'(\alpha)^{-1}\| \le \frac{1}{1 - p_1(e_0)}.$$

Then, turning to the expression of $z_0 - \alpha$ *we have*

$$\|z_0 - \alpha\| \le \frac{1}{2(1 - p(\|x_0 - \alpha\|))}\left[2L\int_0^1 \|\alpha + t(x_0 - \alpha) - x_0\|\|x_0 - \alpha\|dt + L\|x_0 - y_0\|\|x_0 - \alpha\|\right]$$
$$\le \frac{1}{2(1-p(\|x_0-\alpha\|))}\left[2L\frac{\|x_0-\alpha\|^2}{2} + L(\|x_0 - \alpha\| + \|y_0 - \alpha\|)\|x_0 - \alpha\|\right]$$
$$\le \frac{1}{2(1-p(e_0))}[Le_0 + L(e_0 + g_1(e_0)e_0)]e_0$$
$$\le \frac{Le_0(2+g_1(e_0))}{2(1-p(e_0))}e_0 = g_2(e_0)e_0$$

(14)

Where

$$g_2 t = \frac{Lt(2+g_1(t))}{2(1-p_1(t))}.$$

So, we consider $s_2(t) = g_2(t) - 1$ *having that* $s_2(0) = -1$ *and* $s_2(r_1) > 0$. *Therefore,* $s_2(t)$ *has least one root in* $(0,\ r_1)$ *and let* r_2 *be the smallest one. Therefore,* $0\ < r_2 < r_1$ *and*

$$0 \le g_2(t)1 \forall t \in [0,\ r_2),$$

(15)

Then by using (14) and (15), we get

$$\|z_0 - \alpha\| \le g_2(\|x_0 - \alpha\|)\|x_0 - \alpha\| < \|x_0 - \alpha\|.$$

Again using (4) for $k = 0$, we get

$$x_1 - \alpha = z_0 - \alpha - G'(y_0)^{-1}G'(\alpha)G'(\alpha)^{-1}G(z_0)$$

Since $y_0 \in D$ and using (6), we get

$$\|I - G'(\alpha)^{-1}G'(y_0)\| \le \|G'(\alpha)^{-1}(G'(\alpha) - G'(y_0))\|$$
$$\le L_0\|y_0 - \alpha\| \le L_{0^{g_1}}(e_0)e_0 = p_2(e_0) < 1$$

Where

$$p_2(t) = L_{0^{g_1}}(t)t.$$

Then $\exists(G'(\alpha)^{-1}G'(y_0))^{-1}$ and

$$\|G'(y_0)^{-1}G'(\alpha)\| \le \frac{1}{1-p_2(e_0)}.$$

Therefore, by using lemma 1, we get

$$\|x_1 - \alpha\| \le \|z_0 - \alpha\| + \frac{1}{1-p_2(e_0)}(1 + L_0\|z_0 - \alpha\|)\|z_0 - \alpha\|$$
$$= \left(1 + \frac{1+}{1-p_2(e_0)}\right)g_2(e_0)e_0 = g_3(e_0)e_0$$

(16)

Where

$$g_3(t) = (t) = \left(1 + \frac{1+L_0 g_2(t)t}{1-p_2(t)}\right)g_2(t)$$

Consider $s_3(t) = g_3(t) - 1$. Then, $s_3(0) = -1$ and $s_3\ (r_2) > 0$. Therefore, $s_3\ (t)$ has at least one root in $(0, r_2$) and let r_3 the smallest one. Therefore, $0 < r_3 < r_2$

$$0 \le g_3(t) \le 1 \ \forall t \in [0,\ r_3), \tag{17}$$

then by using (16) and (17), we get

$$\| x_1 - \alpha \| \le g_3(\| x_0 - \alpha \|) \| x_0 - \alpha \| < \| x_0 - \alpha \| < \eta$$

Thus, Theorem 1 holds for $k = 0$. Changing x_0, y_0, z_0 and x_1 by x_k, y_k, z_k, x_{k+1}, *we* get the inequalities (8)-(10) for all $k \ge 0$. Since, $\|x_{k+1} - \alpha\| \le \|x_k - \alpha\| < r_3$, this gives $x_{k+1} \in B(\alpha, r_3)$. Also $g_3(t)$ is an increasing function in $[0,\ r_3)$, since $g_3'(t) > 0$ for all $t \in [0, r_3)$. Thus, we get

$$\begin{aligned} \| x_{k+1} - \alpha \| &\le g_3(e_0) \| x_k - \alpha \| \le g_3(e_0) g_3(e_0) \| x_{k-1} - \alpha \| \\ &\le g_3(e_0)^2 g_3(e_0) \| x_{k-2} - \alpha \| \le \ldots \le g_3(e_0)^{k+1} \| x_0 - \alpha \| \end{aligned}$$

Therefore, $\lim_{k \to \infty} x_k = \alpha \ \ as \ g_3(t) < 1$

For getting the uniqueness ball for root α, let $\beta \in B(\alpha, R)$ be such that $F(\beta) = 0$ and $\beta \ne \alpha$. Consider

$$P = \int_0^1 G'(\beta + t(\alpha - \beta))dt$$

.

Using (6), we get

$$\| G'(\alpha)^{-1}(P - G'(\alpha)) \| \le \int_0^1 L_0 \|\beta + t(\alpha - \beta) - \alpha\| dt \le \frac{L_0}{2} \|\alpha - \beta\| = \frac{L_0}{2} R < 1$$

,

therefore, by Banach Lemma, P^{-1} exists. Then,

$$0 = G(\alpha) - G(\beta) = P(\alpha - \beta),$$

we obtain $\alpha = \beta$.

2.1. Numerical examples

In this subsection, we consider numerical examples to demonstrate the applicability of our work. Moreover, we compare our results with the local convergence of a modified Halley-Like method (2) and Chebyshev-Halley-type methods (3) respectively.

Example 1. *Let* $X = Y = \mathrm{R}^3$, $D = \overline{U}(0,1)$. *Define G on D for* $v = (x,y,z)$ *by*

$$G(v) = \left(e^x - 1, \frac{e-1}{2} y^2 + y, z \right)$$

.

Clearly, $\alpha = (0,0,0)$, $G'(\alpha) = G'(\alpha)^{-1} = diag\{1,1,1\}$, $L_0 = e - 1$, and $L = e$. Then, we have

$$r_3 = 0.13125 < r_2 = 0.21657 < r_1 = 0.32495.$$

Example 2. *Consider the system of nonlinear equations*

$$2x_1 - \frac{1}{9}x_1^2 - x_2 = 0,$$

$$-x_1 + 2x_2 - \frac{1}{9}x_2^2 = 0$$

The associated nonlinear operator $F : R^2 \to R^2$ *is given by*

$$\mathsf{G}(x_1,x_2)=\begin{pmatrix} {}_1(x_1,x_2)\\ {}_2(x_1,x_2)\end{pmatrix}$$

Where $\mathsf{G}_1(x_1,x_2)=2x_1-\frac{1}{9}x_1^2-x_2$ *and* $\mathsf{G}_2(x_1,x_2)=-x_1+2x_2-\frac{1}{9}x_2^2$.

Clearly $\alpha = (9,9)^T$ is a solution of above nonlinear system and for all $(x,y) \in R^2$ we have:

$$\|\mathsf{G}'(\alpha)^{-1}(\ \ '(x)-\mathsf{G}'(y))\|=\frac{2}{9}\|x-y\|$$
$$\|\mathsf{G}'(\alpha)^{-1}(\ \ '(x)-\mathsf{G}'(\alpha))\|=\frac{2}{9}\|x-\alpha\|$$

.

Taking $L_0=\frac{2}{9}$ and $L=\frac{2}{9}$, we get

$$r_3 = 1.44284 < r_2 = 2.25000 < r_1 = 3.00000.$$

Example 3. *Consider the nonlinear Hammerstein type integral equation given by*

$$\mathsf{G}(x(s))=x(s)-5\int_0^1 s\,t\,x(t)^3dt, \tag{18}$$

with $x(s)$ *in* $C\,[0,1]$.

Clearly $\alpha = 0$. Taking $L_0 = 7.5$ and $L = 15$, we get

$$r_3 = 0.02481 < r_2 = 0.04185 < r_1 = 0.06667.$$

Now, we compare our results with the local convergence of a modified Halley-Like method (2) and Chebyshev-Halley-type methods (3) respectively. The value of parameters used by these methods are listed in Table 1. The radius of a convergence ball of a fifth order method (4) is compared with method (2) and method (3) in Table 2. We can observe that the larger radius of convergence ball is obtained by our approach.

Table 1: Values of parameters

Examples	a	γ	η
1	1.0125	0.3	0.03
2	1	1/9	2/9
3	1	0.575	0.003

Table 2: Comparison of radius of a ball

Examples	Method (4)	Method (2)	Method (3)
1	0.13125	0.02726	0.00892
2	1.44284	0.55264	0.06989
3	0.02481	0.00709	0.00755

3. The derivative free method and its local convergence analysis

In this section our purpose is to complete the study of iterative method (4), when we use adequate approximation of the derivatives by divided differences. So, now the aim is to obtain the local convergence study in this case.

In order to obtain derivative free iterative methods we approximate derivatives by divided differences. That is an operator $[x, y; G]$ verifying

$$[x, y; G]\,(x-y) = G(x) - G(y), \text{ for all } x,y \in D$$

and if G is Frechet differentiable at´ $\alpha \in D$ then $[\alpha, \alpha'; G] = G'(\alpha)$. One can see different approximations of divided differences in [18, 19].

We consider the derivative free iterative method given for $k = 0,1,2,...$ by

$$\begin{aligned} y_k &= x_k - [x_k, x_k + G(x_k); f]^{-1} G(x_k) \\ z_k &= x_k - 2\left([x_k x_k + G(x_k); G] + [y_k, y_k + G(y_k); G] \right)^{-1} G(x_k) \\ x_{k+1} &= z_k - [y_k, y_k + G(y_k); G]^{-1} G(z_k), \end{aligned} \tag{19}$$

where x_0 is the starting point.

We use the following assumptions for setting the local convergence study in this case. Let $K_0 > 0$, $K > 0$ and for all $x,y,u,v \in D$, we have $G(\alpha) = 0$, $G'(\alpha)^{-1} \neq 0$, in D, moreover

$$\|G'(\alpha)^{-1}([x,y;G] - [u,v;G])\| \leq K(\|x-u\| + \|y-v\|), \tag{20}$$

$$\|G'(\alpha)^{-1}([x,y;G] - [\alpha,\alpha;G])\| \leq K_0(\|x-\alpha\| + \|y-\alpha\|), \tag{21}$$

$$\|G(x) - G(\alpha)\| \leq L\|x-\alpha\|. \tag{22}$$

The next result describes the local convergence theorem for the derivative free iterative method (19)

Theorem 2. *Let F the nonlinear operator satisfying assumptions* (20), (21) *and* (22). *Then, the sequence* $\{x_{k+1}\}$ *generated by* (19) *is well defined for any starting point* $x_0 \in B(\alpha,\rho_3)$ *and converges to* α, *where* ρ_3 *is the smallest positive root of function* q_3. *Also, we obtain the following inequalities for* $k \geq 0$ *:*

$$\|y_k - \alpha\| \leq h_1(\|x_k - \alpha\|)\|x_k - \alpha\| < \|x_k - \alpha\| < \rho_3, \tag{23}$$

$$\|z_k - \alpha\| \leq h_2(\|x_k - \alpha\|)\|x_k - \alpha\| < \|x_k - \alpha\| < \rho_3, \tag{24}$$

$$\|x_{k+1} - \alpha\| \leq h_3(\|x_k - \alpha\|)\|x_k - \alpha\| < \|x_k - \alpha\| < \rho_3, \tag{25}$$

where, h_1, h_2, h_3, q_3 *are auxiliary functions defined in the proof and* ρ_3 *is the smallest root of* $q_3(t)$.

Moreover, if there exists a $R_1 \in \left[\rho_3, \frac{1}{k_0}\right)$ *such that* $\overline{B}(\alpha,R_1) \subseteq D$, *then* α *is the unique solution in* $\overline{B}(\alpha,R_1)$

Proof. Since $x_0 \in D$ *and using (21), we get*

$$\begin{aligned} \|G'(\alpha)^{-1}([x_0, x_0 + G(x_0); G] - [\alpha, \alpha; f])\| &\leq K_0(\|x_0 - \alpha\| + \|x_0 + G(x_0) - \alpha) \\ &\leq K_0(\|x_0 - \alpha\| + (1+L)\|x_0 - \alpha\|) \\ &= K_0(2+L)\|x_0 - \alpha\| < 1, \end{aligned}$$

for $\|x_0 - \alpha\| < \frac{1}{K_0(2+L)}$. Therefore, by Banach Lemma, $[x_0, x_0 + G(x_0); G]^{-1}$ exists and

$$\|[x_0, x_0 + G(x_0); G]^{-1} G'(\alpha)\| \leq \frac{1}{1 - K_0(2+L)\|x_0 - \alpha\|} \quad . \tag{26}$$

Thus, y_0 is well defined. Using (19) for $k = 0$, we get

$$y_0-\alpha = x_0-\alpha-[x_0,x_0+G(x_0);G]^{-1}G(x_0)$$
$$=[x_0,x_0+G(x_0);G]^{-1}G'(\alpha)G'(\alpha)^{-1}([x_0,x_0+G(x_0);G]-[x_0,\alpha;G])(x_0-\alpha)$$

Using (20), we have

$$\|G'(\alpha)^{-1}([x_0,x_0+G(x_0);G]-[x_0,\alpha;G])\| \le K(\|x_0+G(x_0-\alpha\|) \le K(1+L)\,\|x_0-\alpha\|.$$

Therefore,

$$\begin{aligned}\|y_0-\alpha\| &\le \left(\frac{K(1+L)\|x_0-\alpha\|}{1-K_0(2+L)\|x_0-\alpha\|}\right)\|x_0-\alpha\| \\ &= h_1(e_0)e_0,\end{aligned} \tag{27}$$

where, $h_1(t)=\frac{K(1+L)t}{1-K_0(2+L)t}$ and $e_0 = \mathrm{k}x_0-\alpha\mathrm{k}$.

Consider the function $q_1(t) = h_1(t) - 1$. Then, $q_1(0) = -1$ and $q_1(\frac{1}{K_0(2+L)}) \to +\infty$. Therefore, $q_1(t)$ has atleast one root in $\left(0, \frac{1}{K_0(2+L)}\right)$ and let ρ_1 be such a smallest root. Therefore, $0<\rho_1<\frac{1}{K_0(2+L)}$, and

$$0 \le h_1(t) < 1 \ \forall t \in [0, \rho_1). \tag{28}$$

Using (27) and (28), we get

$$\|y_0-\alpha\| \le h_1(\|x_0-\alpha\|)\,\|x_0-\alpha\| < \|x_0-\alpha\|.$$

Again using (19) for $k = 0$, we get

$$\begin{aligned} z_0-\alpha &= x_0-\alpha-2\big([x_0,x_0+G(x_0);G]+[y_0,y_0+G(y_0);G]\big)^{-1}G(x_0) \\ &= \big([x_0,x_0+\ (x_0);G]+[y_0,y_0+G(y_0);\ ]\big)^{-1}G'(\alpha)G'(\alpha)^{-1}\Big([x_0,x_0+G(x_0);G] \\ &+ [y_0,y_0+\ (y_0);G]-2[x_0,\alpha;G]\Big)(x_0-\alpha). \end{aligned}$$

Now we study the existence and bound for the product ,

$$[x_0,x_0+\ (x_0);G]+[y_0,y_0+G(y_0);G]\big)^{-1}G'(\alpha)$$

.

Then we observe that

$$\begin{aligned} &\frac{1}{2}G'(\alpha)^{-1}\Big(\big([x_0,x_0+G(x_0);G]+[y_0,y_0+G(y_0);F]\big)-2G'(\alpha)\Big) \\ &= I-\underbrace{\frac{1}{2}G'(\alpha)^{-1}[([x_0,x_0+G(x_0);G]+[y_0,y_0+G(y_0);G])]}_{C} = I-C. \end{aligned}$$

So we try to apply Banach Lemma

$$\begin{aligned}\|I-C\| &\leq \frac{1}{2}\Big(\|G'(\alpha)^{-1}([x_0,x_0+G(x_0);G]-G'(\alpha))\|+\|G'(\alpha)^{-1}([y_0,y_0+G(y_0);G]-G'(\alpha))\|\Big)\\ &\leq \frac{K_0}{2}\Big((\|x_0-\alpha\|+\|x_0+G(x_0)-\alpha\|)+(\|y_0-\alpha\|+\|y_0+G(y_0)-\alpha\|)\Big)\\ &\leq \frac{K_0}{2}\Big((2+L)\|x_0-\alpha\|+(2+L)\|y_0-\alpha\|\Big)\\ &\leq \frac{K_0}{2}\Big((2+L)\|x_0-\alpha\|+(2+L)h_1(\|x_0-\alpha\|)\|x_0-\alpha\|\Big)\\ &= \phi_1(e_0),\end{aligned}$$

where $\phi_1(t) = \frac{K_0}{2}(2+L)(1+h_1(t))t$ is an increasing function such that $\varphi_1(0) = 0$ and $\varphi_1(\rho_1) = \frac{K_0}{2}(2+L)\rho_1\,(1+h_1(\rho_1)) = \frac{K_0}{2}(2+L)\rho_1 < 1$ and so we have

$$\|C^{-1}\| = \Big\|2\Big([x_0,x_0+G(x_0);G]+[y_0,y_0+G(y_0);G]\Big)^{-1}G'(\alpha)\Big\| \leq \frac{1}{1-\phi_1(e_0)}$$

.

Therefore,

$$\begin{aligned}\|z_0-\alpha\| &\leq \frac{K}{2(1-\phi_1(\|x_0-\alpha\|))}\Big(\|x_0+G(x_0)-\alpha\|+\|y_0-x_0\|+\|y_0+G(y_0)-\alpha\|\Big)\|x_0-\alpha\|\\ &\leq \frac{K}{2(1-\phi_1(e_0))}\Big((1+L)e_0+(1+h_1(e_0))e_0+(1+L)h_1(e_0)e_0\Big)e_0\\ &\leq \frac{K}{2(1-\phi_1(e_0))}\Big((2+L)(1+h_1(e_0))e_0\Big)e_0\\ &= h_2(e_0)e_0,\end{aligned} \quad (29)$$

where

$$h_2(t) = \frac{K}{2(1-\phi_1(t))}\Big((2+L)(1+h_1(t))t\Big).$$

Consider $q_2(t) = h_2(t)-1$. Then, $q_2(0) = -1$ and $q_2(\rho_1) = \frac{K\rho_1(2+L)}{1-K_0(2+L)\rho_1} > 0$. Therefore, $q_2(t)$ has atleast one root in $(0,\rho_1)$ and let ρ_2 be such a smallest root. Therefore, $0 < \rho_2 < \rho_1$ and

$$0 \leq h_2(t) \leq 1 \;\forall t \in [0,\rho_2). \quad (30)$$

Using (29) and (30), we get

$$\| z_0-\alpha \| \leq h_2\big(\| x_0-\alpha \|\big)\| x_0-\alpha \| < \| x_0-\alpha \|$$

Taking $k = 0$ in (19), we get

$$x_1-\alpha = z_0-\alpha-[y_0,y_0+G(y_0);G]^{-1}\, G'(\alpha)\, G'(\alpha)^{-1}G(z_0) \qquad .$$

Since $y_0 \in D$, we have

$$\begin{aligned}\|G'(\alpha)^{-1}([y_0,y_0+G(y_0);G]-[\alpha,\alpha;G])\| &\leq K_0(\|y_0-\alpha\|+\|y_0+G(y_0)-\alpha\|\\ &\leq K_0(\|y_0-\alpha\|+(1+L)\|y_0-\alpha\|)\\ &\leq K_0(2+L)h_1(e_0)e_0 = \phi_2(e_0) < 1,\end{aligned}$$

where $\varphi_2(t) = K_0(2+L)h_1(t)t$. Thus, by Banach Lemma, we have

$$\|[y_0,y_0+G(y_0);G]^{-1}G'(\alpha)\| \leq \frac{1}{1-\phi_2(e_0)}.$$

Therefore,

$$\begin{aligned}\|x_1-\alpha\| &\le \|[y_0, y_0+G\ (y_0);\ \ ]^{-1} G'(\alpha)\|\| G'(\alpha)^{-1}([y_0, y_0+G\ (y_0); G]-[z_0,\alpha;G\ ])\|\|z_0-\alpha\| \\ &\le \frac{K(\|y_0-z_0\|+\|y_0+G\ (y_0)-\alpha\|)}{1-\phi_2(e_0)}\|z_0-\alpha\|.\end{aligned}$$

As

$$\|y_0-z_0\| \le \|y_0-\alpha\|+\|z_0-\alpha\| \le (h_1(t)+h_2(t))\|x_0-\alpha\|.$$

$$\begin{aligned}\|x_1-\alpha\| &= \frac{K\Big((h_1(e_0)+h_2(e_0))\,e_0+(1+L)h_1(e_0)e_0\Big)}{1-\phi_2(e_0)}h_2(e_0)\|x_0-\alpha\| \\ &= h_3(e_0)e_0. \qquad (31)\end{aligned}$$

where

$$h_3(t)=\Big(\frac{K\big((2+L)h_1(t)t+h_2(t)t\big)}{1-\phi_2(t)}\Big)h_2(t).$$

Consider $q_3(t) = h_3(t) - 1$. Then, $q_3(0) = -1$ and $q_3(\rho_2) > 0$. Therefore, $q_3(t)$ has at least one root in $(0,\rho_2)$ and let ρ_3 be such a smallest root. Therefore, $0 < \rho_3 < \rho_2$, and

$$0 \le h_3(t) \le 1\ \forall t \in [0,\rho_3). \qquad (32)$$

Using (31) and (32), we get

$$\|x_1-\alpha\| \le h_3(\|x_0-\alpha\|)\|x_0-\alpha\| < \|x_0-\alpha\|$$

Thus, theorem holds for $k = 0$. Changing x_0,y_0,z_0 and x_1 by x_k,y_k,z_k,x_{k+1}, we get the inequalities (23)-(25) for all $k \ge 0$. Since, $\|x_{k+1}-\alpha\| \le \|x_k-\alpha\| < r_3$, this gives $x_{k+1} \in B(\alpha,\rho_3)$. Also $h_3(t)$ is an increasing function in $[0,\rho_3)$, since $h_3(t) > 0$ for all $t \in [0,\rho_3)$. Thus, we get

$$\begin{aligned}\|x_{k+1}-\alpha\| &\le h_3(e_0)\|x_k-\alpha\| \le h_3(e_0)h_3(e_0)\|x_{k-1}-\alpha\| \\ &\le h_3(e_0)^2h_3(e_0)\|x_{k-2}-\alpha\| \le \cdots \le h_3(e_0)^{k+1}\|x_0-\alpha\|\end{aligned}$$

Therefore, $\lim_{k\to\infty} x_k = \alpha$ as $h_3(t) < 1$.

—

For uniqueness part, let $P_1 = [\alpha,\beta;G]$ where $G(\beta) = 0$ and $\beta \in B(\alpha,R_1)$. Thus, we have

$$\|G'(\alpha)^{-1}(P_1-G'(\alpha))\| \le K_0(\|\ \alpha-\alpha\ \|+\|\ \beta-\alpha\ \|) \le K_0R_1 < 1,$$

therefore, by Banach Lemma, P_1^{-1} exists. Then,

$$0 = G(\alpha) - G(\beta) = P_1(\alpha - \beta),$$

we obtain $\alpha = \beta$.

3.1. Numerical examples

In this subsection, we consider numerical examples to demonstrate the applicability of our work. Moreover, we compare our results with the local convergence of a modified Halley-Like method (2) and Chebyshev-Halley-type methods (3) respectively.

Example 4. *Let* $X = Y = \mathrm{R}$, $D = (-1,1)$. *Define F on D by*

$$G(x) = e^x - 1.$$

$$\alpha = 0,\ G'(\alpha) = G'(\alpha)^{-1} = 1,\ K_0 = \frac{e-1}{2}, \text{ and } L = e.$$

Clearly,
Then, we have

$$r_3 = 0.100343 < r_2 = 0.101834 < r_1 = 0.109801.$$

Example 5. *Let $X = Y = \mathbb{R}$, $D = (-1,1)$. Define G on D by*

$$G(x) = x^2 - 1.$$

Clearly, $\alpha = 1,\ K_0 = K = \frac{1}{2}$, and $L = 2$. Then, we have

$$r_3 = 0.265055 < r_2 = 0.267949 < r_1 = 0.285714.$$

Example 6.*Consider the nonlinear Hammerstein type integral equation given by*

$$G\,(x(s)) = x(s) - 5\int_0^1 s\,t\,x(t)^3 dt, \tag{33}$$

with $x(s)$ in $C[0,1]$.
Clearly $\alpha = 0$. Taking $K_0 = 3.75$, $K = 7.5$ and $L = 8.5$ we get

$$r_3 = 0.008636 < r_2 = 0.008708 < r_1 = 0.009039.$$

The value of parameters are listed in Table 1. The radius of a convergence ball of a derivative free fifth order method (19) is compared with the existing methods and showed in Table 3. We can observe that except in example 5, all other examples larger radius of convergence ball is obtained by our approach. In example 5, we observe that the larger radius of convergence is obtained as compared to the Method (3).

Table 3: Comparison of radius of a ball

Examples	Method (19)	Method (2)	Method (3)
4	0.100343	0.02726	0.00892
5	0.265055	0.55264	0.06989
6	0.008636	0.00709	0.00755

4. Conclusions

This study includes the corresponding study when we consider the derivative free method obtained by approximating the derivatives by divided difference, getting a complete analysis of this iterative method. In this paper, we discussed the local convergence of two fifth order iterative methods for solving nonlinear equations in Banach spaces. For the purposes of this analysis, it is assumed that the first order Frechet's derivative meets the Lipschitz continuity requirement in the first case and a similar condition in the derivative free technique, which uses only the derivative at the precise solution. Finally, various numerical examples were worked out and the radii of convergence were calculated for each method. Additionally, we have contrasted these outcomes with those of other methodologies and found that our outcomes are more effective.

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