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Split Restrained Geodetic Number of a Graph

Ashalatha K S

Assistant Professor, Department of Mathematics,Vedavathi Government First Grade College, Hiriyur, Chitradurga District-577598.

Abstract

Let 'G' be a graph. If $u, v \in V$, then a u-v geodetic of G is the shortest path between u and v. The closed interval I[u, v] consists of all vertices lying in some u-v geodetic of G . For $S \subseteq V(G)$ the set I[S] is the union of all sets I [u, v] for $u, v \in S$. A set S is a geodetic set of G if I[S]=V(G). The cardinality of minimum geodetic set of G is the geodetic number of G, denoted by g(G). A set S of vertices of a graph G is a split geodetic set if S is a geodetic set and $\langle V - S \rangle$ is disconnected, split geodetic number $g_s(G)$ of G is the minimum cardinality of a split geodetic set of G. In this paper I study split restrained geodetic number of a graph. A set S of vertices of a graph G is a split restrained geodetic set if S is a geodetic set and the subgraph $\langle V - S \rangle$ is disconnected with no isolated vertices. The minimum cardinality of a split restrained geodetic set of G is the split restrained geodetic number of G and is denoted by $g_{sr}(G)$. The split restrained geodetic numbers of some standard graphs are determined and also obtain the split restrained geodetic number in the Cartesian product of graphs.

Keywords: Geodetic set, Geodetic number, Split geodetic set, Split geodetic number, Split Restrained Geodetic set, Split Restrained Geodetic number.

1. Introduction

In this paper, we follow the notations of [4]. The graphs considered here have at least one component which is not complete or at least two nontrivial components.

The distance d (u, v) between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. It is well known that this distance is a metric on the vertex set V(G). For a vertex v of G, the eccentricity e(v) is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is radius, rad G, and the maximum eccentricity is the diameter, diam G. A u-v path of length d(u, v) is called a u-v geodesic. We define I [u, v] to the set of all vertices lying on some u-v geodesic of G and for a nonempty subset S of V(G), $I[S] = \cup_{u,v \in S} I[u, v]$. A set S of vertices of G is called a geodetic set in G if I[S]=V(G), and a geodetic set of minimum cardinality is a minimum geodetic set. The cardinality of a minimum geodetic set in G is called the geodetic number of G, and we denote it by g(G). The geodetic number of a graph was introduced in [6,7] and further studied in [2,8,4].

A geodetic set S of a graph G= (V, E) is a split geodetic set if the induced subgraph $\langle V - S \rangle$ is disconnected. The split geodetic number $g_s(G)$ of G is the minimum cardinality of a split geodetic set. The split geodetic number was introduced and studied in [9].

For any undefined term in this paper, see [3] and [4].

2. Preliminary Notes

We need the following results to prove further results.

Theorem 2.1 [2] Every geodetic set of a graph contains its extreme vertices.

Theorem 2.2 [2] For any tree T with k end-edges, g(T)=k.

Theorem 2.3 [2] For any path P_n with n vertices, $g(P_n) = 2$.

Theorem 2.4 [2] For cycle C_n of order $n \geq 3$, $g(C_n) = \begin{cases} 2, \text{if n is even} \\ 3, \text{if n is odd} \end{cases}$

Theorem 2.5 [2] If G is a nontrivial connected graph, then $g(G) \leq g(G \times K_2)$.

3. Main Results

Proposition 3.1 For any graph G, $g(G) \leq g_s(G) \leq g_{sr}(G)$.

Theorem 3.2 For cycle C_n of order $n \geq 6$, $g_{sr}(C_n) = \begin{cases} 2 & \text{if n is even} \\ 3 & \text{if n is odd} \end{cases}$

Proof: Let $n \geq 6$, we have the following cases.

Case 1: Let n be even. Consider $\{v_1, v_2, \dots, v_{2n}, v_1\}$ be a cycle with 2n vertices and let $S = \{v_i, v_j\}$ be a split restrained geodetic set of C_{2n}. For any two antipodal vertices v_i and v_j, the shortest $v_i - v_j$ path includes all the vertices of C_{2n}. Clearly $I[S] = V(C_{2n})$. Also $x, y \in V - S$, V-S is disconnected with no isolated vertices. Hence $g_{sr}(C_{2n}) = 2$.

Case 2: Let n be odd. Consider $\{v_1, v_2, \dots, v_{2n+1}, v_1\}$ be a cycle with 2n+1 vertices and let $S = \{v_i, v_j, v_{j+1}\}$ be a split restrained geodetic set of C_{2n+1}, where $d(v_i, v_j) = diam(C_{2n+1})$. By theorem 2.4 and proposition 3.1, no two vertices of S forma split restrained geodetic set, so there exists a split restrained geodetic set contains three vertices. Clearly $I[S] = V(C_{2n+1})$. Also V-S is disconnected with no isolated vertices. Hence $g_{sr}(C_{2n+1}) = 3$.

Theorem 3.3 For any path P_n, $n \geq 7$, $g_{sr}(P_n) = 3$.

Proof: Let $S = \{v_1, v_n, v_i\}$ be a split restrained geodetic set of P_n, where v_1, v_n are the two end vertices. By theorem 2.3, $v_1 - v_n$ path includes all the vertices of P_n and v_i is a cut vertex which forms two connected components and for any $x, y \in V - S$, such that V-S is disconnected with no isolated vertices. Thus $g_{sr}(P_n) = 3$.

Theorem 3.4 Let T be a caterpillar that has at least five internal vertices, if T has k end-vertices, then $g_{sr}(T) = k + 1$.

Proof: Let $S = \{v_1, v_2, \dots, v_k\}$ be the set containing end-vertices of T and itself a geodetic set of T such that $I[S\} = V(T)$. Let $\{u_1, u_2, \dots, u_l) \subset V - S$. Now $S' = S \cup \{u_i\}$ is a split restrained geodetic set of T, where $u_i \in V - S$ which is a cut vertex which forms connected components. Consider $P = \{v_1, v_2, \dots, v_k\}$ be a set of end-vertices of T such that$|P| < |S'|$ is a geodetic set but V-P is connected, so P is not a split restrained geodetic set. Again $Q = \{u_1, u_2, \dots, u_l\}$ be set of internal vertices of T such that $|Q| < |S'|$ is not a geodetic set. Hence it is clear that S' is a minimum split restrained geodetic set of T. Also $x, y \in V - S'$, such that $V - S'$ is disconnected with no isolated vertices. Hence $g_{sr}(T) = k + 1$.

Theorem 3.5 For any Tadpole graph $g_{sr}(T_{m,n}) = \begin{cases} 3 & \text{if m is even}, n \geq 3, m > 4 \\ 4 & \text{if m is odd}, n \geq 3, m > 3 \end{cases}$

Proof: Tadpole graph is a special type of graph consisting of cycle graph of m vertices and a path graph of n vertices connected with a bridge.

$V = \{c_1, c_2, \dots, c_m\}$ are the vertices of C_m and $U = \{p_1, p_2, \dots, p_n\}$ are the vertices of P_n. $W = V \cup U$ are the vertices of tadpole graph.

For $n \geq 3$, we have the following cases.

Case 1: m is even and $m > 4$

Let $S = \{p_n, c_i, c_j\}, i < j \leq m$ be a split restrained geodetic set of $T_{m,n}$, where P_n is the end vertex of $T_{m,n}$ and c_i, c_j are the antipodal vertices C_n. Suppose $S' = \{p_n, c_i\}, |S'| < |S|$, which is a geodetic set and $V - S'$ is connected. Hence S is a minimum split restrained geodetic set. Also for all $x, y \in V - S$, it follows that V-S is disconnected with no isolated vertices. Thus $g_{sr}(T_{m,n}) = 3$.

Case 2: m is odd and $m > 4$.

Let $S = \{p_n, c_i, c_{i+1}\}$ is a geodetic set such that $I[S] = V(T_{m,n})$, where $d(p_n, c_i) = d(p_n, c_{i+1}) = diam(T_{m,n})$. Let $\{c_1, c_2, \dots, c_j, p_1, p_2, \dots, p_{n-1}\} \subset V - S$. Now $S' = S \cup \{p_l\}$ or $S \cup \{c_k\}$ is a split restrained geodetic set of $T_{m,n}$, where p_l or $c_k \in V - S$ which is a cut vertex which forms connected components. Hence it is clear that S' is a minimum split restrained geodetic set of $T_{m,n}$. Also $x, y \in V - S'$ it follows that $V - S'$ is disconnected with no isolated vertices. Hence $g_{sr}(T_{m,n}) = 4$.

Theorem 3.6 For Banana tree graph $g_{sr}(B_{n,k}) = n(k - 2) + 1$.

Proof: Let $S = \{v_1, v_2, \dots, v_{nk-2n}\}$ be the set containing end-vertices of $B_{n,k}$ and itself a geodetic set of $B_{n,k}$, such that $I[S] = V(B_{n,k})$. Let $S' = S \cup \{u\}$ is a split restrained geodetic set of $B_{n,k}$, where $u \in V - S$ is a root vertex. Consider $P = \{v_1, v_2, \dots, v_{nk-2n}\}$ be a set of end vertices of $B_{n,k}$ such that $|P| < |S'|$ is a geodetic set but $V - P$ is connected, so P is not a split restrained geodetic set. Again $Q = \{u, u_1, u_2, \dots, u_l\}$ be set of internal vertices of $B_{n,k}$ such that $|Q| < |S'|$ is not a geodetic set. Hence it is clear that S' is a minimum split restrained geodetic set of $B_{n,k}$. Also $x, y \in V - S'$, such that $V - S'$ is disconnected with no isolated vertices. Hence $g_{sr}(B_{n,k}) = n(k - 2) + 1$.

4. Adding an End-Edge

For an edge $e = (u, v)$ of a graph G with $\deg(u) = 1$ and $\deg(v) > 1$, we call e an end-edge and u an end–vertex.

Theorem 4.1 For the Helm graph H_n $n \geq 6$, $g_{sr}(H_n) = n + 3$.

Proof: Let Helm graph H_n is a graph obtained from the wheel graph by attaching an end-edge at each vertex of the n-cycle of the wheel.

Let $V(H_n) = \{x, u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ where $\deg(x) = n$, $\deg(v_i) = 1$ and $\deg(u_i) = 4$ for each $i \in \{1,2, \dots, n\}$.

Let $S = \{v_1, v_2, \dots, v_n, x\}$ be the set of n-end vertices of H_n and a vertex of degree n is a geodetic set of H_n such that $I[S] = V(H_n)$. Let $\{u_1, u_2, \dots, u_n\} \subset V - S$. Now $S' = S \cup \{u_i, u_j\}$ is a split restrained geodetic set of H_n, where $u_i, u_j \in V - S$ which are cut vertices forms connected components. Consider $P = \{v_1, v_2, \dots, v_{k,} x\}$ be a set of end vertices and a vertex of degree n of H_n such that $|P| < |S'|$ is a geodetic set but $V - P$ is connected, so P is not a split restrained geodetic set, again $Q = \{u_1, u_2, \dots, u_n\}$ be the vertices of the cycle of H_n, such that $|Q| < |S'|$ is not a geodetic set. Hence it is clear that S' is the minimum split restrained geodetic set of H_n. Also $x, y \in V - S'$, it follows that $V - S'$ is disconnected with no isolated vertices. Hence $g_{sr}(H_n) = n + 3$.

Theorem 4.2 Let G' be the graph obtained by adding an end-edge (u_i, v_i), i=1, 2…, n to each vertex of cycle $C_n = G$ for $n \geq 6$, such that $u_i \in G, v_i \notin G$, then $g_{sr}(G') = n + 2$.
Proof: Let $G = C_n = \{u_1, u_2, \dots, u_n, u_1\}$ be a cycle with n vertices. Let G' be the graph obtained by adding an end-edge (u_i, v_i), $i = 1,2, \dots, n$ to each vertex of G such that $u_i \in G, v_i \notin G$. Clearly $X = \{v_1, v_2, \dots, v_n\}$ is the n number of end-vertices of G'. Let $S = X \cup \{u_i, u_j\}$ be a split restrained geodetic set of G', where u_i, u_j are cut vertices which forms connected components. Thus $I[S] = V(G')$. Also $x, y \in V - S$, such that $V - S$ is disconnected with no isolated vertices, thus $g_{sr}(G') = n + 2$.

Theorem 4.3 Let G' be the graph obtained by adding k end-edges $\{(u, v_1), (u, v_2), \dots, (u, v_k)\}$ to a cycle $C_n = G$ of order $n \geq 6$, with $u \in G$ and $\{v_1, v_2, \dots, v_k\} \notin G$.

Then $g_{sr}(G') = \begin{cases} k+2 & \text{for even cycle} \\ k+3 & \text{for odd cycle} \end{cases}$

Proof: Let $G = C_n = \{u_1, u_2, \dots, u_n, u_1\}$ be a cycle with n vertices and let G' be the graph obtained from $G = C_n$ by adding k end-edges $\{(u, v_1), (u, v_2), \dots, (u, v_k)\}$ for fixed $u \in G$ and $\{v_1, v_2, \dots, v_k\} \notin G$. We have the following cases.
Case 1: Let $G = C_{2n}$,n> 2. Consider $S = \{v_1, v_2, \dots, v_k\} \cup \{u_i\}$, for any vertex u_i of G. Now $S' = S \cup \{u\}$ be a split restrained geodetic set,$\{v_1, v_2, \dots, v_k\}$ are the end –vertices of G' and u, u_i are antipodal vertices of G, thus $I[S'] = V(G')$. Consider $P = \{v_1, v_2, \dots, v_k\}$ as a set of end-vertices such that $|P| < |S'|$ is not a geodetic set, that is for some vertex $u_i \in V(G)$, $u_i \notin I[P]$. If $P = S$, then P is not split restrained geodetic set. Thus S' is the minimum split restrained geodetic set. Then $V - S'$ is an induced subgraph which has more than one connected component. Thus $g_{sr}(G') = k + 2$.
Case 2: Let $G = C_{2n+1}, n > 3$. Consider $S = \{v_1, v_2, \dots, v_k\} \cup \{u_i, u_{i+1}\}$ for any adjacent vertices $u_i, u_{i+1} \in G$. Now $S' = S \cup \{u\}$ be a split restrained geodetic set, such that $\{v_1, v_2, \dots, v_k\}$ are the end-vertices of G' and $d(u, u_i) = d(u, u_{i+1}) = diam(G)$. Thus $I[S'] = V(G')$. For any $x, y \in V - S'$,it follows that $V - S'$ is disconnected with no isolated vertices. Thus $g_{sr}(G') = k + 3$.

5. Cartesian Product

The Cartesian product of the graphs H_1 and H_2, written as $H_1 \times H_2$, is the graph with vertex set $V(H_1) \times V(H_2)$, two vertices (u_1, u_2) and (v_1, v_2) being adjacent in $H_1 \times H_2$ if and only if either $u_1 = v_1$ and $(u_2, v_2) \in E(H_2)$, or $u_2 = v_2$ and $(u_1, v_1) \in E(H_1)$.

Theorem 5.1 For any cycle C_n of order n, $g_{sr}(K_2 \times C_n) = \begin{cases} 4 & \text{if n is even} \\ 5 & \text{if n is odd}, n > 3 \end{cases}$

Proof: Consider $G = C_n$, Let $K_2 \times C_n$ be a graph formed from two copies G_1 and G_2 of G. Let $V = \{v_1, v_2, \dots, v_n\}$ be the vertices of G_1 and $W = \{w_1, w_2, \dots, w_n\}$ be the vertices of G_2 $U = V \cup W$. We have the following cases.
Case 1: Let n be even. Consider $S = \{v_i, w_j\}$ is a geodetic set of $K_2 \times G$ such that $v_i - w_j$ path is equal to $diam(K_2 \times G)$, which includes all the vertices of $K_2 \times G$. Let $S' = \{v_i, w_i, v_j, w_j\}$, where $(v_i, w_i), (v_j, w_j) \in E(K_2 \times G)$. Now $U - S'$ is an induced subgraph which is disconnected but no isolated vertices. Hence $g_{sr}(K_2 \times G) = 4$.
Case 2: Let n be odd. Consider $S = \{v_i, w_j, v_k\}$ is the geodetic set of $K_2 \times G$ such that $d(v_i, w_j) = diam(K_2 \times G) = d(w_j, v_k)$. Thus $I[S] = U(K_2 \times G)$. Let $S' = \{v_i, w_i, v_j, w_j, v_k\}$, where

$(v_i, w_i), (v_j, w_j) \in E(K_2 \times G)$. Now $U - S'$ is an induced subgraph which is disconnected with no isolated vertices. Hence $g_{sr}(K_2 \times G) = 5$.

Theorem 5.2 For any path P_n of order n>3, $g_{sr}(K_2 \times P_n) = 4$.
Proof: Consider $G = P_n$. Let $K_2 \times P_n$ be a graph formed from two copies G_1 and G_2 of G. Let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertices of G_1, $W = \{w_1, w_2, \ldots w_n\}$ be the vertices of G_2 and $U = V \cup W$. Let $n > 3$. If $S = \{v_i, w_j,\}$ is a geodetic set of $K_2 \times P_n$ where $d(v_i, w_j) = diam(K_2 \times P_n)$. Let $S' = S \cup \{v_{i+1}, w_{j+1}\}$, Now $U - S'$ is an induced subgraph which is disconnected with no isolated vertices. Hence $g_{sr}(K_2 \times P_n) = 4$.

Theorem 5.3 For any complete graph of order $n > 3$, $g_{sr}(K_2 \times K_n) = n$.
Proof: Let G_1 and G_2 be disjoint copies of $G = K_n, n > 3$. Let $V = \{v_1, v_2, \ldots, v_n\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ be the vertex sets of G_1 and G_2, respectively. Let S be a minimum geodetic set of $K_2 \times K_n$. Without loss of generality, we may assume that $v_1 \in S$. Since $d(v_1, w_j) = 2 = diam(K_2 \times K_n)$ for each $j = \{2,3, \ldots, n\}, \{w_j | 2 \leq j \leq n\} \subseteq S$. So $g(K_2 \times K_n) \geq n$, and thus $g_{sr}(K_2 \times K_n) \geq n$. Since $\{v_1\} \cup \{w_j | 2 \leq j \leq n\}$ forms a split restrained geodetic set of $K_2 \times K_n$, $g_{sr}(K_2 \times K_n) \leq n$. Thus $g_{sr}(K_2 \times K_n) = n$.

6. References

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