International Journal for Multidisciplinary Research (IJFMR)

E-ISSN: 2582-2160 ● Website: www.ijfmr.com ● Email: editor@ijfmr.com

The Amalgamation Formulae and Iterations of Mono-Digit Pedigree

Humphrey Takunda Muchapireyi

Midlands State University

Abstract

In this paper we present equations to generate the concatenation of natural numbers $A \in \mathbb{N}$. The main ingredient of our study is the innocuous repunit of a number R_A ; and in this study we introduce three amalgamation operators: ↕ for the constant, ↑ for the incremental and ↓ for the detrimental cases of concatenation according to a variable arithmetic progression. Our method of amalgamation demonstrates the generation of common number sequences, as well as how any two different natural numbers can be joined and alternated, depending on which of the numbers should be displayed first. We also introduce an operator $\Omega(A)$ based on repunits and whose dynamics yields two constants H_o and H_e which we calculate. Amongst the few corollaries from this conjecture is an identity stating that the product of the repunits of those irrational numbers $R_{H_o}R_{H_e}$is equal to unity.

Keywords: Concatenation, Repunit, Amalgamation, Sequences

1. Introduction

In this paper, we introduce equations and operators using the repunits and palindromic numbers [1]. Regarding the composition of natural numbers, mathematicians from the Greeks to the great Gauss ascribed the role of atoms on the number line to prime numbers. Hypotheses such as the fundamental theorem of arithmetic [2] hint at the perception that primes constitute all natural numbers and that it is through factorization that every natural number is uniquely created. We assert that every natural number $A \in \mathbb{N}$ can be concatenated with units of itself or other numbers; and manipulating the repunit function R_A provides a means to do so seamlessly. In this paper we also present a general equation for the expanded form of repunit summands expressed as combinations of products; as well as presenting the modified repunit function (conjectured by Witno and others) which is quite useful in our method of concatenation; consequently producing some of the beautiful sequences such as even sequences with a visible arithmetic progression. Smarandache and others have studied various concatenated patterns for the purposes of finding infinite sequences of prime numbers. In this paper we devise new *amalgamation* functions to describe the constant, incremental and detrimental cases of numerical concatenation. Based on R_A again, we also describe a *omega* operator over a natural number $\Omega(A)$ which we study, resulting in the derivation of two irrational constants H_o and H_e which we approximate to a few decimal places by calculating the omega of some random natural numbers repeatedly, for very large numbers of iterations to mimic even and odd limit to infinity.

1

2. On the expanded form of repunit summands and the modified repunit

2.1. Proposition The expanded form for the repunit of the sum of numbers $a_1, a_2, a_3 \dots a_n \in \mathbb{N}$ up to $n = 4$:

$$R_{(a_1+a_2)} = R_{a_1} + R_{a_2} + 9R_{a_1}R_{a_2} \tag{2.11}$$

$$R_{(a_1+a_2+a_3)} = R_{a_1} + R_{a_2} + R_{a_3} + 9R_{a_1}R_{a_2} + 9R_{a_2}R_{a_3} + 9R_{a_1}R_{a_3} + 9^2R_{a_1}R_{a_2}R_{a_3}$$

$$R_{(a_1+a_2+a_3+a_4)} = R_{a_1} + R_{a_2} + R_{a_3} + R_{a_4} + 9R_{a_1}R_{a_2} + 9R_{a_2}R_{a_3} + 9R_{a_1}R_{a_3} + 9R_{a_1}R_{a_4} + 9R_{a_2}R_{a_4} + 9R_{a_4}R_{a_3} + 9^2R_{a_1}R_{a_2}R_{a_3} + 9^2R_{a_1}R_{a_3}R_{a_4} + 9^2R_{a_1}R_{a_2}R_{a_4} + 9^2R_{a_2}R_{a_3}R_{a_4} + 9^3R_{a_1}R_{a_2}R_{a_3}R_{a_4}.$$

In terms of combinations of terms, disregarding order,

$$R_{(a_1+a_2)} = 9^0\{\text{sum of (2 choose 1) terms of the product of 1 unique factor combination}\} + 9^1\{\text{sum of (2 choose 2) terms of the product of 2 unique factor combinations}\}$$

$$R_{(a_1+a_2+a_3)} = 9^0\{\text{sum of (3 choose 1) terms of the product of 1 unique factor combination}\} + 9^1\{\text{sum of (3 choose 2) terms of the product of 2 unique factor combinations}\} + 9^2\{\text{sum of (3 choose 3) terms of the product of 3 unique factor combinations}\}$$

$$R_{(a_1+a_2+a_3+a_4)} = 9^0\{\text{sum of (4 choose 1) terms of the product of 1 unique factor combination}\} + 9^1\{\text{sum of (4 choose 2) terms of the product of 2 unique factor combinations}\} + 9^2\{\text{sum of (4 choose 3) terms of the product of 3 unique factor combinations}\} + 9^3\{\text{sum of (4 choose 4) terms of the product of 4 unique factor combinations}\}.$$

In compact form, we assign $\prod [R_{a_\mu}]$ as the product $R_{a_j} R_{a_k} R_{a_l} \dots$ containing h factors exhausting all the possible unique combinations from $\binom{n}{h}$, the general repunit of summands equation for $h, i, l, n \in \mathbb{N}$ is:

$$R_{[\sum_{i=1}^{n}(a_i)]} = \sum_{h=1}^{n} \{9^{h-1} \sum_{\text{terms with } h \text{ factors } = 1}^{\text{terms with } h \text{ factors } = \binom{n}{h}} [\prod R_{a_\mu}]\}. \tag{2.12}$$

Remark: All $R_{[\sum_{i=1}^{n}(a_i)]}$ can be condensed into repunits of the sum of only two numbers y and y since the sum $a_1 + a_2 + a_3, \dots a_n = a_x + a_y$; and from 2.11 by the associative property of addition for any $x, y \in \mathbb{N}$:

$$R_{(x+y)} = R_{(y+x)} = R_y + (1 + 9R_y)R_x = R_x + (1 + 9R_x)R_y. \tag{2.13}$$

2.2. Proposition There exists a variation of repunits consisting of a series of repdigit 0's which we denote by $z \in \mathbb{N}$ between two consecutive 1's in a repunit. Witno defined this function as $P_{k,n} = \sum_{i=0}^{n-1}\{10^{ki}\}$ [3]. However for our purposes, we maintain the repunit symbol R_A instead of $P_{k,n}$ for the number of ones, as well as assigning the subscript $z \in \mathbb{N}$ to represent the number of zeros. For instance, the modified repunit $R_{2|0}$ remains $R_2 = 11$, however $R_{4|2} = 1001001001$. In general, the modified $R_{A|z}$ for $A, z \in \mathbb{N}$ is given by:

$$R_{A|z} = \frac{R_{\{A(z+1)\}}}{R_{(z+1)}}. \tag{2.21}$$

3. On the concatenation of *constant* sub-numbers

When Carl Gauss anecdotally added the numbers 1 up to 100, his result was the product 101(50), which is the sum 5050 [4]. From proposition 2.2, the coefficient of 50 in the product can be thought of as the modified repunit we defined in 2.21 i.e. $R_{2|1} = 101$. As such, some of the equations in our method of concatenation contain the modified repunit. Looking at the sum 5050 through a new lens, Gauss' sum is also the concatenation of two values of the same sub-number 50.

A vital aspect of our method is to make sure that the introduction of any new notation or operators must not compromise the arithmetic and algebraic integrity of numbers. For instance, even though the number 987987987 exhibits a pattern of 3 concatenated units of 987, the numerical value of 987987987 i.e. nine hundred eighty-seven million nine hundred eighty-seven thousand nine hundred eighty-seven ought to be preserved after the operation.

3.1. Definition We introduce an amalgamation function $\updownarrow(A, n)$ to concatenate n terms whose value are the number A_1, resulting in the concatenated number $A_1A_1A_1 \ldots A_n \in \mathbb{N}$. We also define q_{A_1} as the number of digits in A_1, q_{A_2} as the number of digits in A_2 and so forth.

3.2. Example. Considering a natural sub-number say 307 which has 3 digits, we decide to concatenate that number with 3 other sub-numbers (making $n = 4$) with that same value and digit count; then the result 307307307307 should also be a natural number. Similarly, the concatenation of 3 units of the natural sub-number 26976 should yield a larger natural number 269762697626976.

3.3. Corollary The amalgam function for the concatenation of n units of the same (constant) sub-number A_1 can be calculated using the formula, where $A_1, q_A, n \in \mathbb{N}$:

$$\updownarrow(A_1, n) = A_1 \frac{R_{(nq_{A_1})}}{R_{q_{A_1}}}. \quad (3.31)$$

3.4. Corollary For the concatenation of an amalgam (itself an amalgam that concatenates $a \in \mathbb{R}$ units of the same sub-number A_1) to produce a higher amalgam of $b \in \mathbb{R}$ units of that same A_1, then the operation requires $\frac{b}{a}$ units of the smaller amalgam. Mathematically, for $a < b \in \mathbb{R}$

$$\updownarrow\left\{\updownarrow(A_1, a), \frac{b}{a}\right\} = \updownarrow(A_1, b). \quad (3.41)$$

3.5. Corollary The number of digits of an amalgam described by $\updownarrow(A_1, a)$, where $A_1, q_A \in \mathbb{N}$ and $a \in \mathbb{R}$, is

$$q_{\updownarrow(A_1,a)} = a q_{\updownarrow(A_1,1)} = a q_{A_1}. \quad (3.51)$$

4. The Amalgamation Formula for *arithmetic-progressive* concatenation

4.1. Claim Based on Gauss' sum 5050 again [4], we consider other numbers 5051, 505560 or 50525456. These are concatenated sub-numbers obeying an A.P with 1st term $A_1 = 50$, variable common difference d and an n^{th} term. Assuming $q_{A_1} = q_{A_2} = \ldots q_{A_n} \in \mathbb{N}$, it follows that for $A_1, n \in \mathbb{N}$

$$d(q_A - 1) = A_n - A_{n-1}.$$

4.2. Conjecture The concatenation of sub-numbers obeying an arithmetic progression of first term A_1, a common difference d and a final n^{th} term, $A_1A_2A_3 \ldots A_n$ is given in two parts: first the amalgamation function $\uparrow(A, n, d)$ for the incremental case to the n^{th} term; and $\downarrow(A, n, d)$ denoting the corresponding detrimental. The incremental & detrimental amalgam equations in **repunit** form for $A_1, q_A, n, d, k \in \mathbb{N}$,

$$\uparrow(A_1, n, d) = \frac{R_{(nq_{A_1})}}{R_{q_{A_1}}}\left[A_1 - \frac{R_{(2q_{A_1})}}{R_{q_{A_1}}}\right] + \sum_{k=2}^{n+1}\left\{\frac{R_{(kq_{A_1})}}{R_{q_{A_1}}}\right\} + (d-1)\sum_{k=1}^{n-1}\left\{\frac{R_{(kq_{A_1})}}{R_{q_{A_1}}}\right\} \quad (4.21)$$

$$\downarrow(A_1, n, d) = \frac{R_{(nq_{A_1})}}{R_{q_{A_1}}}\left[A_1 + \frac{R_{(2q_{A_1})}}{R_{q_{A_1}}}\right] - \sum_{k=2}^{n+1}\left\{\frac{R_{(kq_{A_1})}}{R_{q_{A_1}}}\right\} - (d-1)\sum_{k=1}^{n-1}\left\{\frac{R_{(kq_{A_1})}}{R_{q_{A_1}}}\right\}. \quad (4.22)$$

These are their equivalents in **omega functional** form (introduced in section 6) for all $A_1, q_A, n, d, k \in \mathbb{N}$:

$$\uparrow(A_1,n,d) = \left[\frac{n\Omega q_{A_1}}{\Omega(nq_{A_1})}\right]\left[A_1 - \frac{2\Omega q_{A_1}}{\Omega(2q_{A_1})}\right] + \sum_{k=2}^{n+1}\left\{\frac{k\Omega q_{A_1}}{\Omega(kq_{A_1})}\right\} + (d-1)\sum_{k=1}^{n-1}\left\{\frac{k\Omega q_{A_1}}{\Omega(kq_{A_1})}\right\} \quad (4.23)$$

$$\downarrow(A_1,n,d) = \left[\frac{n\Omega q_{A_1}}{\Omega(nq_{A_1})}\right]\left[A_1 + \frac{2\Omega q_{A_1}}{\Omega(2q_{A_1})}\right] - \sum_{k=2}^{n+1}\left\{\frac{k\Omega q_{A_1}}{\Omega(kq_{A_1})}\right\} - (d-1)\sum_{k=1}^{n-1}\left\{\frac{k\Omega q_{A_1}}{\Omega(kq_{A_1})}\right\}. \quad (4.24)$$

Combining equations 3.31 with 2.21, we arrive at the relation:

$$\updownarrow(A_1,n) = A_1 R_{[n|(q_{A_1}-1)]}. \quad (4.25)$$

Hence the equivalents of 4.21 & 4.22 in **modified repunit** form for all $A_1, q_A, n, d, k \in \mathbb{N}$:

$$\uparrow(A_1,n,d) = R_{[n|(q_{A_1}-1)]}\left[A_1 - R_{[2|(q_{A_1}-1)]}\right] + \sum_{k=2}^{n+1}\{R_{[k|(q_{A_1}-1)]}\} + (d-1)\sum_{k=1}^{n-1}\{R_{[k|(q_{A_1}-1)]}\}$$
(4.26)

$$\downarrow(A_1,n,d) = R_{[n|(q_{A_1}-1)]}\left[A_1 + R_{[2|(q_{A_1}-1)]}\right] - \sum_{k=2}^{n+1}\{R_{[k|(q_{A_1}-1)]}\} - (d-1)\sum_{k=1}^{n-1}\{R_{[k|(q_{A_1}-1)]}\}.$$
(4.27)

4.3. Examples applying equations 4.26 and 4.27.

$$\uparrow(305,4,6) = R_{[4|2]}[305 - R_{[2|2]}] + \sum_{k=2}^{5}\{R_{[k|2]}\} + 5\sum_{k=1}^{3}\{R_{[k|2]}\}$$

= 1001001001(305 − 1001) + (1001 + 1001001 + 1001001001 + 1001001001001 + 5(1 + 1001 + 1001001)
= 305311317323.

$$\downarrow(7659,3,2) = R_{[3|3]}[7659 + R_{[2|3]}] - \sum_{k=2}^{4}\{R_{[k|3]}\} - 1\sum_{k=1}^{2}\{R_{[k|3]}\}$$

= 100010001 (7659 + 10001) - (10001 + 100010001 + 1000100010001) - 1(1 + 10001)
= 765976577655.

Here we have shown the incremental concatenation of 4 units starting with the sub-number 305 with each term increasing by a value of 6, as well as the detrimental concatenation of 3 sub-numbers starting with 305 and each term decreasing by a value of 2.

Note. When we add the incremental amalgam 4.21 and detrimental amalgam 4.22 amalgams of A_1 for exactly the same *nth* term, but removing the common difference i.e. when $d = 0$.

$$\uparrow(A_1,n,0) + \downarrow(A_1,n,0) = \frac{R_{(nq_{A_1})}}{R_{q_{A_1}}}\left[A_1 + \frac{R_{(2q_{A_1})}}{R_{q_{A_1}}}\right] + \sum_{k=2}^{n+1}\left\{\frac{R_{(kq_{A_1})}}{R_{q_{A_1}}}\right\} + (d-1)\sum_{k=1}^{n-1}\left\{\frac{R_{(kq_{A_1})}}{R_{q_{A_1}}}\right\} +$$
$$\frac{R_{(nq_{A_1})}}{R_{q_{A_1}}}\left[A_1 + \frac{R_{(2q_{A_1})}}{R_{q_{A_1}}}\right] - \sum_{k=2}^{n+1}\left\{\frac{R_{(kq_{A_1})}}{R_{q_{A_1}}}\right\} - (d-1)\sum_{k=1}^{n-1}\left\{\frac{R_{(kq_{A_1})}}{R_{q_{A_1}}}\right\}$$
$$= 2\left\{A_1\frac{R_{(nq_{A_1})}}{R_{q_{A_1}}}\right\}$$

And from 3.31, $$A_1\frac{R_{(nq_{A_1})}}{R_{q_{A_1}}} = \updownarrow(A_1,n)$$

$$\therefore \quad \uparrow(A_1,n,0) + \downarrow(A_1,n,0) = 2\{\updownarrow(A_1,n)\}. \quad (4.31)$$

This validates the reasoning that the *incremental* and *detrimental* amalgams reduce to exactly the same function (when there is no arithmetic spacing between sub-numbers i.e. $d = 0$); and that function is also the *constant* amalgam for all values of $A_1, q_A, n, k \in \mathbb{N}$.

4.4 Corollary It follows that terms of common sequences like the Back Concatenated Even Sequence from A038396 in OEIS [5] can be generated by the detrimental, for $A_1, q_A, n, k \in \mathbb{N}$ only for $q_{A_1} = q_{A_2} = \dots q_{A_n}$

$$\downarrow (A_1, n, 2) = R_{[n|(q_{A_1}-1)]}\left[A_1 + R_{[2|(q_{A_1}-1)]}\right] - \sum_{k=2}^{n+1}\{R_{[k|(q_{A_1}-1)]}\} - \sum_{k=1}^{n-1}\{R_{[k|(q_{A_1}-1)]}\} \quad (4.41)$$

Example For instance, the first 12 terms of the sixteenth A038396 can be generated by the detrimental:

$$\downarrow (32,12,2) = 323028262422201816141210.$$

It is worth noting that the remaining 4 lower terms cannot be properly calculated by this formula as they have $q_A = 1$ instead of 2. In general, any values of $A_1, n \in \mathbb{N}$ can be varied to generate similar even sequences.

5. On the concatenation of any two quantities and alternating sequences

An important question would be, just how can one literally join any two *different* numbers $A, B \in \mathbb{N}$ by concatenation? In this section we show this to be a special case of either the arithmetic-progressive incremental or detrimental amalgams (depending on which of the two numbers is required to come first).

Corollary 5.1 The concatenation of any two quantities $A, B \in \mathbb{N}$ where $A < B$ and $q_A = q_B = q$ can be given in two cases in this section.

Case 1 Where the smaller number A must come first, the incremental amalgam containing $n = 2$ terms, and a common difference $d = B - A$; then the concatenation of $A, B, q \in \mathbb{N}$ is calculated as follows,

$$\uparrow (A, 2, B - A) = \frac{\mathrm{R}_{(2q)}}{\mathrm{R}_q}\left[A - \frac{\mathrm{R}_{(2q)}}{\mathrm{R}_q}\right] + \sum_{k=2}^{3}\{\frac{\mathrm{R}_{(kq)}}{\mathrm{R}_q}\} + (B - A - 1)\sum_{k=1}^{1}\{\frac{\mathrm{R}_{(kq)}}{\mathrm{R}_q}\}$$

$$= B - A - 1 + \frac{\mathrm{R}_{(3q)}}{\mathrm{R}_q} + \frac{\mathrm{R}_{(2q)}}{\mathrm{R}_q}\left[1 + A - \frac{\mathrm{R}_{(2q)}}{\mathrm{R}_q}\right]. \quad (5.11)$$

Case 2 Where it is required for the larger number B to come first, the equation for the concatenation of two numbers A and B is given by the following equation; as before $n = 2$, $d = B - A$ and $A, B, q \in \mathbb{N}$.

$$\downarrow (B, 2, B - A) = \frac{\mathrm{R}_{(2q)}}{\mathrm{R}_q}\left[A + \frac{\mathrm{R}_{(2q)}}{\mathrm{R}_q}\right] - \sum_{k=2}^{3}\{\frac{\mathrm{R}_{(kq)}}{\mathrm{R}_q}\} - (B - A - 1)\sum_{k=1}^{1}\{\frac{\mathrm{R}_{(kq)}}{\mathrm{R}_q}\}$$

$$= A - B + 1 - \frac{\mathrm{R}_{(3q)}}{\mathrm{R}_q} + \frac{\mathrm{R}_{(2q)}}{\mathrm{R}_q}\left[B - 1 + \frac{\mathrm{R}_{(2q)}}{\mathrm{R}_q}\right]. \quad (5.12)$$

Example Suppose we wish to concatenate two natural numbers 456 and 699 such that the smaller number 456 comes first in the result. We use equation 5.11 to compute the incremental amalgam:

$$\uparrow (456,2,243) = 699 - 456 - 1 + \frac{\mathrm{R}_9}{\mathrm{R}_3} + \frac{\mathrm{R}_6}{\mathrm{R}_3}(1 + 456 - \frac{\mathrm{R}_6}{\mathrm{R}_3})$$

= 456699.

Example Likewise if we want to concatenate two other natural numbers $A = 1070$ and $B = 5000$ so that this time, the larger number 5000 comes first in the result. We use equation 5.12 to compute the detrimental:

$$\downarrow (5000,2,1070) = 1070 - 5000 + 1 - \frac{\mathrm{R}_{12}}{\mathrm{R}_4} + \frac{\mathrm{R}_8}{\mathrm{R}_4}[5000 - 1 + \frac{\mathrm{R}_8}{\mathrm{R}_4}]$$

=50001070.

Hence for all $A, B, q \in \mathbb{N}$, 5.11 and 5.12 can compute the concatenation of any two different natural numbers.

Corollary 5.2 The constant amalgamation of the resultant amalgams from 5.11 or 5.12 up to m values are equivalent to the alternating constant amalgams for each case, of A and B for m values of both A and B, i.e.

$$\updownarrow\{\uparrow(A,2,B-A),m\} = \{\uparrow(A,2,B-A)\}\frac{\mathrm{R}_{\{mq_{\uparrow(A,2,B-A)}\}}}{\mathrm{R}_{q_{\uparrow(A,2,B-A)}}}.$$

But from 3.51, $q_{\uparrow(A,2,B-A)} = 2q$

$$\updownarrow\{\uparrow(A,2,B-A),m\} = \frac{\mathrm{R}_{(2mq)}}{\mathrm{R}_{2\mathrm{q}}}\uparrow(A,2,B-A)$$

$$= \frac{\mathrm{R}_{(2mq)}}{\mathrm{R}_{2\mathrm{q}}}\left\{B-A-1+\frac{\mathrm{R}_{(3q)}}{\mathrm{R}_{\mathrm{q}}}+\frac{\mathrm{R}_{(2q)}}{\mathrm{R}_{\mathrm{q}}}\left[1+A-\frac{\mathrm{R}_{(2q)}}{\mathrm{R}_{\mathrm{q}}}\right]\right\}. \quad (5.21)$$

Which is the alternating constant amalgamation of m values of the incremental amalgam $\uparrow(A,2,B-A)$. Similarly, for m units of the detrimental amalgam $\downarrow(B,2,B-A)$, the alternating constant amalgam is given by:

$$\updownarrow\{\downarrow(B,2,B-A),m\} = \{\downarrow(B,2,B-A)\}\frac{\mathrm{R}_{\{mq_{\downarrow(B,2,B-A)}\}}}{\mathrm{R}_{q_{\downarrow(B,2,B-A)}}}$$

Also from 5.33$\downarrow(B,2,B-A) = 2q$

$$\updownarrow\{\downarrow(B,2,B-A),m\} = \frac{\mathrm{R}_{(2mq)}}{\mathrm{R}_{2\mathrm{q}}}\downarrow(B,2,B-A)$$

$$= \frac{\mathrm{R}_{(2mq)}}{\mathrm{R}_{2\mathrm{q}}}\left\{A-B+1-\frac{\mathrm{R}_{(3q)}}{\mathrm{R}_{\mathrm{q}}}+\frac{\mathrm{R}_{(2q)}}{\mathrm{R}_{\mathrm{q}}}\left[B-1+\frac{\mathrm{R}_{(2q)}}{\mathrm{R}_{\mathrm{q}}}\right]\right\}. \quad (5.22)$$

Example Using the result of our example for $\uparrow(456,2,243)$, whereby we want the smaller number 456 to appear before the larger 699; we hereby show that the constant amalgamation of 5 units of 456699 is equal to 5 units of the numbers 456 and 699 alternating between each other by concatenation. Using equation 5.21:

$$\updownarrow\{\uparrow(456,2,243),5\} = \frac{\mathrm{R}_{30}}{\mathrm{R}_6}\uparrow(456,2,243) = 456699456699456699456699456699$$

Example Using our other example for the detrimental concatenation of two numbers 5000 and 1070 wherein we require the larger 5000 to appear first in the sequence $\downarrow(5000,2,3930)$, we also show that the constant amalgamation of 4 units of 50001070 is equal to 4 units of the numbers 5000 and 1070 concatenated such that they alternate between each other up to 4 terms each. Using equation 5.22, it follows that:

$$\updownarrow\{\downarrow(5000,2,3930),4\} = \frac{\mathrm{R}_{32}}{\mathrm{R}_8}\uparrow(5000,2,3930) = 50001070500010705000107050001070$$

6. On iterations of mono-digit pedigree

6.1. Definition We assign an omega function as the ratio of a number $A \in \mathbb{N}$ to its repunit R_A:

$$\Omega(A) = \frac{9A}{10^A-1}. \quad (6.11)$$

It is worth noting that $\Omega(0)$ is undefined for $A \in \mathbb{R}$, however the omega of negative values of A is positive

$$\Omega(-A) = 10^A\Omega(A)$$
$$= 9A + \Omega(A). \quad (6.12)$$

6.2. Proposition Extending 6.1 to the omega operation on A for higher orders, we can perform further iterations $v = 2,3,4 \ldots \in \mathbb{N}$ such that the output of the function is fed back into the input, from $A = 1,2 \ldots \in \mathbb{N}$,

$$A = \Omega^0(A)$$

$$\frac{A}{R_A} = \Omega^1(A)$$

$$\frac{(\frac{A}{R_A})}{R_{(\frac{A}{R_A})}} = \Omega^2(A) = \Omega\{\Omega(A)\}$$

$$\frac{\{\frac{A}{R_A R_{(\frac{A}{R_A})}}\}}{R_{\{\frac{A}{R_A R_{(\frac{A}{R_A})}}\}}} = \frac{A}{R_A R_{(\frac{A}{R_A})} R_{\{\frac{A}{R_A R_{(\frac{A}{R_A})}}\}}} = \Omega\{\Omega[\Omega(A)]\} = \Omega^3(A).$$

Continuing this for higher values of $v \in \mathbb{N}$, we hereby define the v^{th}order omega function $\Omega^v(A)$.

6.3. Conjecture The value of the v^{th} order omega function, when the output of the function is fed back into the input, will converge to the irrational number $\boldsymbol{H_o} = \mathbf{0.004526296}$ ... as the number of iterations v tends to odd infinity and $\boldsymbol{H_e} = \mathbf{3.888317383}$ ... as v tends to even infinity, independently of A, for all $v, A \in \mathbb{N}$,

$$\lim_{v\to\infty} \Omega^v A = \begin{cases} H_{e,} \text{ iff } v \equiv 0 \bmod 2 \\ H_{o,} \text{ iff } v \equiv 1 \bmod 2. \end{cases} \quad (6.31)$$

6.4. Corollary It follows that at some point in the infinitude of $v \in \mathbb{N}$, the value of $\Omega^v(A)$ tends to the exact values of H_e and H_o. If $\Omega^v(A)$ converges to H_e first, then automatically the value of the next iterate will be H_o and vice versa. This is analogous to how every Collatz sequence eventually collapses to a value of 1 independently of the initial value fed into the dynamical system [6]. Therefore the following equations hold,

$$\Omega H_o = H_e \quad (6.41)$$

$$\Omega H_e = H_o. \quad (6.42)$$

6.5. Corollaries. Along with 6.41 and 6.42, these equations are identities of H_e and H_o from Conjecture 6.3:

$$10^{H_o+H_e} = 10^{H_o} + 10^{H_e} + 80 \quad (6.51)$$

$$R_{(H_o+H_e)} = R_{H_o} + R_{H_e} + 9 \quad (6.52)$$

$$R_{H_o} = \frac{H_o}{H_e} \quad (6.53)$$

$$R_{H_e} = \frac{H_e}{H_o} \quad (6.54)$$

$$R_{H_o} R_{H_e} = 1. \quad (6.55)$$

7. Conclusion

In this paper we have demonstrated how the simple repunits of natural numbers can yield equations that describe the concatenation of numbers of the same digit count. Our method of concatenation makes use of an amalgam function, which describes the relationship between the initial natural sub-number $A \in \mathbb{N}$, the number of units of the sub-numbers to be concatenated, the digit count of all these sub-numbers (the number of digits must in fact be the same for all sub-numbers in question), and finally the arithmetic spacing between consecutive sub-numbers to be concatenated into the final number pattern. Employing again the repunit of a natural number R_A, we also presented a conjecture that if the ratio of a number to its repunit is presented as a function, and the output is fed back into the input for iterations approaching infinity, the value of that function tends to two irrational constants depending on whether the number of iterations is even or odd. Hence we gave our approximate values for H_o and H_e by numerical computation.

References

1. C. W. Wu, "Pigeonholes and Repunits," *The American Mathematical Monthly*, vol. 121, no. 6, pp. 529-533, 2014.
2. G. H. Hardy and E. M. Wright, "Statement of the fundamental theorem of arithmetic," *" Proof of the Fundamental Theorem of Arithmetic," and" Another Proof of the Fundamental Theorem of Arithmetic,* vol. 1, no. 2.1, p. 3, 1979.
3. Witno, "A Family of Sequences Generating Smith Numbers," *Journal of Integer Sequences,*, vol. 16, no. 13.4.6, p. 3, 2013.
4. Hayes, "Computing Science Gauss's Day of Reckoning," *American Scientist,* vol. 94, no. 3, pp. 200-201, 2005.
5. M. Coman, Sequences of Intergers, Conjectures and New Aerithmetical Tools (Collected Papers), Education Publishing, 2015.
6. T. Tao, "Almost all orbits of the Collatz map attain almost bounded values," *Forum of Mathematics, Pi. Cambridge University Press*, vol. 10, p. 1, 2022.
7. H. Beiler, Recreations in the Theory of Numbers: The Queen of Mathematics Entertains, Courier Corporation, 1964.