Families of Disjoint Sets Colouring Technique and Concept of Common and Non Common Vertex

Mansoor ElShiekh Hassan Osman Satti

Associate Professor, Department of Pure Mathematics, Faculty of mathematical Sciences and Informatics, University of Khartoum, Khartoum, Sudan

ABSTRACT:
Families of disjoint sets colouring technique is trial to generalize all type of colouring, such as edge colouring, vertex colouring, face colouring, sets colouring (partition of sets into families of disjoint sets). In this paper we introduce the concept of common vertex between two cliques and more than two cliques. We introduce some results related to concept of common and non common vertex, and introduce some results explain the number of minimum colour classes not changed after addition or after removal of non common vertex, if maximum clique number is constant. Also in this paper we introduce concept of quasi non common vertex between two cliques and more than two cliques and we introduce some results related to this concept.

KEYWORDS: families of disjoint sets colouring technique, common vertex, non common vertex, quasi non common vertex, proper clique.

I. INTRODUCTION

In paper [2] we introduced families of disjoint sets colouring technique, and published papers [3],[4],[5],[6],[7],[8],[9],[to introduce some results related to this colouring technique.

In paper [10] we introduce the concept of common set to establish some results for families of disjoint sets colouring technique. The concept of common set, common edge and common vertex are essential concept for families of disjoint sets technique, and we have to modify some definitions in previous papers due to this. Paper [11] analogue to paper [10]. In paper [11] we introduce the concept of common edge to establish some results as generalization to some theorems of edge colouring.

This paper analogue to some sections of paper [10] and paper [11], in section three we introduce the concept of common vertex and proper clique, also we introduce some results related to these two concepts. In section four we partition vertices into minimum number of colour classes, it is the same method in paper [11] used to partition edges into minimum number of colour classes. Also this method is same as method used to partition sets into minimum number of families of disjoint sets [10]. In section five we introduce concept of quasi non common vertex between two cliques and between more than two cliques and explain how addition or removal of number of quasi non common vertices not change minimum number of colour classes.

It papers [2],[3],[4],[5],[6],[7],[8] and [9] each concept of edge, vertex and set is indefinite. For families of disjoint sets colouring technique, vertex means common vertex or non common vertex or quasi non common vertex, and so on for edge and set, for this reason we modified some definitions, notations and results in this paper and paper [9] and paper [10]. Also some definitions and results still need modifications, so when you go through papers [2],[3],[4],[5],[6],[7],[8] and [9], you have to put this in considerations.

II.RELATED WORK

In this section we recall some definitions and results related to families of disjoint sets colouring technique. Families of disjoint sets colouring technique is unification of edge colouring, vertex colouring, face colouring and partition of sets, as we can see in Theorem 2.13., Corollary 2.14 Proposition 5.8., and Proposition 5.9.

We try to minimize definitions and concepts related to this technique, through previous papers ([2],[3],[4],[5],[6],[7],[8] and [9]), there are some definitions and concepts not be used through this paper, e.g. Remark 2.3., Remark 2.4., Proposition 2.10. and Definition2.11.

Concept of common set, common edge and common vertex appear too late in paper [10] and paper [11], due to this you will find modifications of definitions in next works, and modifications of definitions in this paper such as Definition 3.10., Definition 3.11. and Definition 3.12.

Comparing between Proposition 2.5. and Proposition 2.6. explain concepts of common edge and non common edge are essential concept for families of disjoint sets technique.

Theorem 2.1. The sum of degrees of vertices equals twice the number of edges [12].

Definition 2.2. A clique of a graph is a maximal complete sub graph. A clique number of a graph G is largest number n such that K_n is a subgraph of G [1].

Remark 2.3. Colouring of families of disjoint sets technique depend on the number n, generally in case of sets n is degree of intersection or n is the degree of set labelled by families of disjoint sets, for colouring edges of the graph n is the largest degree of vertex (maximum number of edges such that any two are adjacent). If minimum number of colours m equal n it is trivial value, if $m=n+1$ it is nontrivial value. For colouring vertices of the graph n is the clique number (maximum number of vertices such that any two are adjacent) [10].

Remark 2.4.. If we take any three vertices of maximum degree in the graph G and if it include a vertex of degree one, then colouring edges of this graph is trivial colouring [7].

Proposition2.5. Let $\chi_1,\chi_2,\chi_3,...,\chi_m$ be m colour classes to colour edges of graph G, and $v_1,v_2,v_3,...,v_w$ be w vertices with maximum degree of vertex equal n, and t_i number of times the colour class χ_i appear at these w vertices, then we have $\sum_{i=1}^{m} t_i = \sum_{j=1}^{w} d(v_j)$ where $1 \le i \le m$ and t_i is even for all i [8].

Proposition 2.6. Let G be a graph of k common and non common edges $e_1,e_2,e_3,...,e_k$ and these k edges intersect at w proper vertices $v_1,v_2,v_3,...,v_w$ (each vertex is intersection of at least two common edges), with maximum degree of vertex equal n, if all edges (common edges and non common edges) partition into m colour classes $\chi_1,\chi_2,\chi_3,...,\chi_m$, labelled by $1,2,3,...,m$, and if t_i be number of times the colour class

χ_i appear at these w vertices, where $1 \le i \le m$, and for all $1 \le j \le w$ then we have $\sum_{l=1}^{k} c_l + \sum_{l=1}^{k_1} c_l = \sum_{j=1}^{w} d(v_j)$, where c_l appearance of edge e_l at w vertices, $1 \le l \le k$, $1 \le l \le k_1$, k_1 number of non common edges [11].
Remark 2.7. In this paper we will labelled a vertex using its degree and edges. If v is vertex of degree three we can used the notation $v=(e_i, e_j, e_k)$ pointing to v as common point and any two of edges e_i, e_j, e_k are adjacent. The notation $v=(i, j, k)$ pointing to colours i, j, k, where i, j, k are integers. If v of degree n we write $v=(e_1, e_2, e_3, ..., e_n)$ for pointing to edges $e_1, e_2, e_3, ..., e_n$, or we can used notation $v=(1,2,3,..., n)$ for pointing to colours (or colour classes) $1,2,3, \ldots, n$ we say v represented (or labelled) by $1,2,3, \ldots, n$ [3].
Remark 2.8. The four colours labeled by 1,2,3,4, and $X_1=(1,2,3)$, $X_2=(1,2,4)$, $X_3=(1,3,4)$ represent three intersections each of degree three. $Y_1=(1,2,4)$, $Y_2=(1,2,3)$, $Y_3=(3,4)$, represent three intersections two of degree three and one of degree two [5].
Remark 2.9. In papers [2] and [3] we used the notations $X=(1,2,3,..., r)$, $v=(1,2,3,..., n)$, $X=(1,2,3,5)$, from this paper we modify these notations, we will write these notations as follows: $X \equiv (1,2,3,..., r)$, $v \equiv (1,2,3,..., n)$, $X \equiv (1,2,3,5)$. If edge e incident with vertices v_1, v_2, we use the notation $e \equiv (v_1, v_2)$ [10].
Proposition 2.10. The number of colour classes used to colour edges of connected graph that is regular of degree two (cycle graph) with even number of vertices, is trivial colouring [6].
Definition2.11. Let G be a graph, m is called local minimum colour classes to colour edges of graph G, at w vertices $v_1, v_2, v_3, ..., v_w$, if we use $m-1$ colour classes to colour edges of graph G, then exists at least two adjacent edges belong to one colour class or satisfy trivial colouring (every colour class appear be w times at be w vertices) [8].
Proposition2.12. If G is a graph consists of one clique with clique number n if we connect this clique to one vertex of degree two, then number of colours to colour these $n+1$ vertices equal n [9].
Theorem 2.13. Maximum number of families of disjoint sets with degree of intersection n is equal $2n-1$[4].
Corollary 2.14. Let $v_1, v_2, v_3, ..., v_w$ be w vertices each with degree of colouring equal n, m be number of different colours at these w vertices, then we have $m \le 2n-1$ [3].

III. COMMON VERTEX AND NONCOMMON VERTEX

In this section we introduce the concept of common vertices between two cliques or between number of cliques more than two, and introduce proper clique and improper clique, and introduce some results related to these concepts.
Definition3.1. Let C_1 be clique of the vertices $v_1, v_2, v_3, ..., v_t$, then C_1 is called improper clique of vertices $v_1, v_2, v_3, ..., v_t$ if there exists a clique C_2 of the vertices $\{u_1, u_2, u_3, ..., u_s\}$ such that $\{v_1, v_2, v_3, ..., v_t\} \subset \{u_1, u_2, u_3, ..., u_s\}$, $t < s$, $C_1 \subset C_2$, and for all i and some j we have $v_i = u_j$, where $C_1 = \{v_1, v_2, v_3, ..., v_t\}$, $C_2 = \{u_1, u_2, u_3, ..., u_s\}$, $1 \le i \le t, 1 \le j \le s$.
Definition3.2. Let C_1 be clique of the vertices $v_1, v_2, v_3, ..., v_t$, then C_1 is called proper clique of vertices $v_1, v_2, v_3, ..., v_t$ if for all C_2, and $C_1 \subset C_2$, then C_2 is not clique for any the vertices $u_1, u_2, u_3, ..., u_s$, such that for all i and some j we have $v_i = u_j$, where $C_1 = \{v_1, v_2, v_3, ..., v_t\}$, $C_2 = \{u_1, u_2, u_3, ..., u_s\}$, $1 \le i \le t, 1 \le j \le s$.

Definition 3.3. In a graph G if each of C_1 and C_2 be proper clique of the vertex v, then v is called common between two cliques C_1 where C_2, if $C_1 \neq C_2$, and each of C_1, C_2 is clique of number of vertices not less than two include the vertex v. A vertex v is called common vertex between t proper cliques $C_1, C_2, C_3, ..., C_t$ if $C_i \neq C_j$, for all i, j where $1 \leq i < j \leq t$ and each one of $C_1, C_2, C_3, ..., C_t$ is clique of number of vertices not less than two and each clique includes v.

Definition 3.4. In a graph G the vertex v is called non common between any number of cliques, if there is only one clique include all vertices adjacent to v or v is isolated vertex. This means a non common vertex, belongs to only one clique.

Suppose k vertices $v_1, v_2, v_3, ..., v_k$ be partition into m colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, labelled by $1,2,3,..., m$, first the colour class χ_i labelled by i where $1 \leq i \leq m$, and each vertex belongs to the colour class χ_i labelled by i. Second for each clique C_j where $1 \leq j \leq w$ and if C_j of degree n it labelled by n labels and each label represent a label of a vertex, see Example 3.5.

Example 3.5. Let G be a graph of nine vertices $v_1, v_2, v_3, ..., v_9$ and three cliques C_1, C_2, C_3, each clique with clique number equal four. C_1 is clique of four vertices v_1, v_2, v_3, v_4, C_2 is clique of four vertices define v_4, v_5, v_6, v_7 if define C_3 is clique of four vertices v_7, v_8, v_9, v_1. We use for cliques notations $C_1 \equiv (v_1, v_2, v_3, v_4)$, $C_2 \equiv (v_4, v_5, v_6, v_7)$, $C_3 \equiv (v_7, v_8, v_9, v_1)$. The vertex v_4 is common vertex between the two proper cliques C_1, C_2 , the vertex v_7 is common vertex between the two proper cliques C_2, C_3 and v_1 is common vertex between the two proper cliques C_3, C_1. The vertices $v_2, v_3, v_5, v_6, v_8, v_9$ are non common vertices. If these nine vertices partition into four colour classes $\chi_1, \chi_2, \chi_3, \chi_4$ where $\chi_1 = \{v_1, v_5\}$, $\chi_2 = \{v_4, v_8\}$, $\chi_3 = \{v_7, v_2\}$ and $\chi_4 = \{v_3, v_6, v_9\}$. For colour classes $\chi_1, \chi_2, \chi_3, \chi_4$ we use labels $1,2,3,4$ respectively and we can write $(1) \equiv \chi_1 = \{v_1, v_5\}$, $(2) \equiv \chi_2 = \{v_4, v_8\}$, $(3) \equiv \chi_3 = \{v_7, v_2\}$ and $(4) \equiv \chi_4 = \{v_3, v_6, v_9\}$. We labelled v_1 by 1, v_5 labelled by 1, v_2 labelled by 3 and v_3 labelled by 4. Also we can write $C_2 \equiv (v_4, v_5, v_6, v_7)$ or $C_2 \equiv (2,1,4,3)$.

Remark 3.6. In this paper for a graph G we mean by clique C any complete sub graph G and we mean by proper sub clique C^* of a clique C sub graph of the graph C such that $C^* \subset C$. By maximum clique number equal n we mean there exists at least one clique of n vertices and all these vertices are common, by common vertex we mean neither non common vertex nor quasi non common vertex, by vertices we mean it can be any one of common vertices, non common vertices and quasi non common vertices.

Definition 3.7. Let C_1 be clique of the vertices $v_1, v_2, v_3, ..., v_t$, then C_1 is called isolate clique of vertices $v_1, v_2, v_3, ..., v_t$ if each one of $v_1, v_2, v_3, ..., v_t$ is non common vertex.

Proposition 3.8. Let G be a graph of k common and non common vertices $v_1, v_2, v_3, ..., v_k$ (common between proper cliques $C_1, C_2, C_3, ..., C_w$), with maximum clique number equal n, if these k vertices and non common vertices partition into m colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, and $v_1, v_2, v_3, ..., v_p$ are vertices (common and non common) belong to one colour class χ_j, if a_i be number of times the vertex v_i appear at these w cliques, $1 \leq i \leq p$, $1 \leq j \leq m$, and t_j be number of times the colour class χ_j appears at these w cliques, then we have $t_j = a_1 + a_2 + a_3 + ... + a_p \leq w$.

Proof: Suppose $v_1, v_2, v_3, ..., v_p$ belong to one colour class χ_j and χ_j labelled by j, then each one of $v_1, v_2, v_3, ..., v_p$ labelled by j and the label j appear $a_1 + a_2 + a_3 + ... + a_p$ times at $C_1, C_2, C_3, ..., C_w$ then whatever we exchange partition of all vertices (common and non common) into m colour classes, and for the vertices $v_1, v_2, v_3, ..., v_p$ belong to one colour class, any two vertices of $v_1, v_2, v_3, ..., v_p$ are nonadjacent, then the label j can not appear twice at any clique, if j appear once at any one of $C_1, C_2, C_3, ..., C_w$ then $t_j = a_1 + a_2 + a_3 + ... + a_p = w$, if j appear once at some of $C_1, C_2, C_3, ..., C_w$ then $t_j = a_1 + a_2 + a_3 + ... + a_p < w$, therefore we have $t_j = a_1 + a_2 + a_3 + ... + a_p \le w$.

Remark 3.9. Dealing with non common vertex is easy than dealing with common vertex. Let C be a proper clique labelled as $C \equiv (1,2,3,..., n)$ the label i represent one colour class, where $1 \le i \le n \le m$, for each i either it represents common vertex belongs to this colour class or non common vertex belongs to this colour class. If it is common vertex, the label i depends on two factor (1) the labels $1, 2, 3, ..., i-1, i+1, i+2, ..., n$, as each label represent a vertex adjacent to the vertex i. (2) t proper cliques, the vertex is common vertex between t cliques , means this common vertex appear at t cliques and with same label. If it is non common vertex, the label i depend only on one factor, the labels $1, 2, 3, ..., i-1, i+1, i+2, ..., n$, as each label represents a vertex adjacent to the vertex i. For this reason when we partition k vertices (common vertices and non common vertices) into m colour classes, to deal with non common vertex is easy than common vertex.

Definition 3.10. Let G be a graph of k vertices $v_1, v_2, v_3, ..., v_k$ (common between proper cliques $C_1, C_2, C_3, ..., C_w$), let these k vertices partition into m colour classes, then these m colour classes are called minimum colour classes, if whatever we try to partition common vertices(among these k vertices) into $m-1$ colour classes, there exist at least two adjacent vertices belong to the same colour class.

Definition 3.11. Let G be a graph of k vertices $v_1, v_2, v_3, ..., v_k$ (common between proper cliques $C_1, C_2, C_3, ..., C_w$), let these k vertices partition into m colour classes, then these m colour classes are called maximum colour classes, if whatever we try to partition common vertices (among these k vertices) into $m+1$ colour classes, there exist two colour classes, such that any two common vertices belong to these two colour classes, are disjoint.

Definition 3.12. Let G be a graph of common and non common vertices $v_1, v_2, v_3, ..., v_k$ (common between proper cliques $C_1, C_2, C_3, ..., C_w$), these k vertices partition into m colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, if w^* be number of proper cliques $C_1^*, C_2^*, C_3^*, C_{w^*}^*$ of common vertices $v_1^*, v_2^*, v_3^*, ..., v_{k^*}^*$ satisfies $w^* + 1 \le t_i + t_j$, for any two colour classes χ_i and χ_j , then the cliques $C_1^*, C_2^*, C_3^*, C_{w^*}^*$ called standard of partition these k^* common vertices into minimum m colour classes, where t_i and t_j be number of times the colour classes χ_i and χ_j appears at these w^* cliques respectively, $V^* \subseteq V$, $\Psi^* \subseteq \Psi$, $k^* \le k$, $w^* \le w$, $V = \{v_1, v_2, v_3, ..., v_k\}$, $V^* = \{v_1^*, v_2^*, v_3^*, ..., v_{k^*}^*\}$, $\Psi = \{C_1, C_2, C_3, ..., C_w\}$, $\Psi^* = \{C_1^*, C_2^*, C_3^*, ..., C_{w^*}^*\}$, $1 \le i < j \le m$.

Remark3.13. You can see Proposition 3.14. , Proposition 3.15. and Proposition 3.16. are same idea of Theorem 2.1. For proof we use definition of clique number and definition of common vertex. The difference between these propositions and Theorem2.1. is that every edge is common between two vertices, and a vertex can be common between any number of cliques.

Proposition 3.14. Let G be a graph of k common vertices $v_1, v_2, v_3, ..., v_k$ (common between proper cliques $C_1, C_2, C_3, ..., C_w$), if these k vertices partition into m colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, for all $1 \le j \le w$ then we have $\sum_{l=1}^{k} a_l = \sum_{j=1}^{w} d(C_j)$, where $1 \le l \le k$ and a_l is appearance of vertex v_l at w cliques.

Proof: Suppose these k vertices partition into m colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, labelled by $1,2,3,..., m$, and let the colour class χ_i labelled by i and each vertex belongs to the colour class χ_i labelled by i. Since each clique C_j labelled by some number of $1,2,3,..., m$, and this number equal $d(C_j)$, the number $d(C_j)$ equal number of vertices of clique C_j, and each vertex v_l labelled by one label of $1,2,3,..., m$, if $v_l \in \chi_i$, and the colour class χ_i contain only of v_l, the label i appears a_l times at w cliques, a_l means the vertex v_l common between a_l cliques. If the colour class χ_i consists only of v_l and v_r, the label i appears $a_l + a_r$ times at w cliques, and we can write $a_l + a_r = t_i$, where t_i be number of times the colour class χ_i appear at these w cliques. If the colour class χ_i consists only of the vertices v_l, v_r, v_s the label i appears $a_l + a_r + a_s$ times at w cliques, and we can write $a_l + a_r + a_s = t_i$, so from definition of clique number and definition of vertex common between some cliques, then we have $\sum_{l=1}^{k} a_l = \sum_{j=1}^{w} d(C_j)$.

Proposition 3.15. Let G be a graph of k common and non common vertices $v_1, v_2, v_3, ..., v_k$ (common between proper cliques $C_1, C_2, C_3, ..., C_w$), if these k vertices partition into m colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, labelled by $1,2,3,..., m$, and for all $1 \le j \le w$ then we have $\sum_{l=1}^{k_1} a_l + \sum_{l=1}^{k_2} b_l = \sum_{j=1}^{w} d(C_j)$, and $k_2 + \sum_{l=1}^{k_1} a_l = \sum_{j=1}^{w} d(C_j)$ where k_1 is number of common vertices, and k_2 is number of non common vertices, a_l, b_l are appearance of common vertex and non common vertex respectively at w cliques.

Proof: Using Proposition 3.14. if $k = k_1 + k_2$ where k_1 is number of common vertices, and k_2 is number of non common vertices, a_l, b_l are appearance of common vertex and non common vertex of common vertex and non common vertex at w cliques, then we have $\sum_{l=1}^{k_1} a_l + \sum_{l=1}^{k_2} b_l = \sum_{j=1}^{w} d(C_j)$. From definition of common vertex and non common vertex at w cliques, we have $\sum_{l=1}^{k_2} b_l = k_2$, therefore $k_2 + \sum_{l=1}^{k_1} a_l = \sum_{j=1}^{w} d(C_j)$.

Proposition 3.16. Let G be a graph of k common and non common vertices $v_1, v_2, v_3, ..., v_k$ (common between proper cliques $C_1, C_2, C_3, ..., C_w$), if these k vertices partition into m colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, labelled by $1,2,3,..., m$, and if t_i be number of times the colour class χ_i appear at these w cliques, where $1 \le i \le m$, and if k_1 is number of common vertices, k_2 is number of non common vertices, a_l, b_l are appearance of common vertex and non common vertex respectively at w cliques, and for all $1 \le j \le w$ then we have $\sum_{i=1}^{m} t_i = \sum_{l=1}^{k_1} a_l + \sum_{l=1}^{k_2} b_l = \sum_{j=1}^{w} d(C_j) \le wn.$

Proof: Using Proposition 3.15. if t_i be number of times the colour class χ_i appear at these w cliques, where $1 \le i \le m$, and since these k vertices partition into m colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, labelled by $1,2,3,..., m$, where $k_1 + k_2 = k$, then $\sum_{i=1}^{m} t_i = \sum_{l=1}^{k_1} a_l + \sum_{l=1}^{k_2} b_l$, and then we have $\sum_{i=1}^{m} t_i = \sum_{l=1}^{k_1} a_l + \sum_{l=1}^{k_2} b_l = \sum_{j=1}^{w} d(C_j) \le wn.$

Remark 3.17. Let G be a graph of k common and non common vertices $v_1, v_2, v_3, ..., v_k$ (common between proper cliques $C_1, C_2, C_3, ..., C_w$), if we add one vertex v_{k+1} to one clique C_j where $d(C_j) = p$ and $1 \le j \le w$ such that after addition $d(C_j) = p+1$, since v_{k+1} belongs only to the clique C_j, then the vertex v_{k+1} is non common vertex.

The following proposition explain the number of minimum colour classes not changed after addition of non common vertex, if clique maximum number not changed.

Proposition 3.18. Let G be a graph of k common and non common vertices $v_1, v_2, v_3, ..., v_k$ (common between proper cliques $C_1, C_2, C_3, ..., C_w$), with maximum clique number equal n, if these k vertices partition into m colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, labelled by $1,2,3,..., m$, if we add only one non common vertex v_{k+1} without changing the maximum clique number n, then these $k+1$ vertices partitioned into m colour classes.

Proof: Let these m colour classes labelled by $1,2,3,..., m$, since after additional of a non common vertex v_{k+1} the proper cliques remains $C_1, C_2, C_3, ..., C_w$, since after addition of non common vertex v_{k+1}, then there is only one clique changed, let that clique be C_j, and $d(C_j) = p$ before addition, after addition C_j becomes C_j^*, we have $d(C^*) = p+1$, where $1 \le j \le w$, $2 \le p \le n-1$, let $C_j \equiv (1,2,3,..., p)$ then after addition of the non common vertex v_{k+1} we labelled C_j^* by $C_j^* \equiv (1,2,3,..., p, q)$ where $p+1 \le q \le m$, and v_{k+1} belongs to colour class χ_q, and since before we add v_{k+1}, $d(C_j) \le n-1$ then v_{k+1} can belongs to one of the following colour classes $\chi_{p+1}, \chi_{p+2}, \chi_{p+3}, ..., \chi_m$. Then these $k+1$ common and non common vertices partitioned into m colour classes.

Corollary 3.19. Let G be a graph of k common and non common vertices $v_1, v_2, v_3, ..., v_k$ (common between proper cliques $C_1, C_2, C_3, ..., C_w$), with maximum clique number equal n, if these k vertices partition into m colour classes $\chi_1, \chi_2, \chi_3, ..., \chi_m$, labelled by $1,2,3,..., m$, if the vertex v_k is non common vertex, and we remove only the vertex v_k without changing the maximum clique number n, then these $k-1$ vertices partitioned into colour classes also equal m.

Proof: Using Proposition 3.18. if we add a non common vertex v_{k+1} without changing the maximum clique number n, then these $k+1$ vertices partitioned into colour classes also equal m. If we remove the non common vertex v_{k+1}, the maximum clique number remains n, then these k vertices was partitioned into colour classes also equal m. Hence removal of a non common vertex, without changing the maximum clique number n, doesn't change number of colour classes.

To introduce simple examples we choose in each of the following two examples all common vertices have same number of appearance (common between same number of cliques).

Example3.20. Let G be a graph of 7 cliques and 14 common vertices, each vertex is common between two cliques, with maximum clique number equal 4, and $\sum_{j=1}^{7} d(C_j) = 28$. First let $m = 5$ and $t_1 + t_2 + t_3 + t_5 = 6 + 6 + 6 + 6 + 4 = 28$. Second let $m = 6$ and $t_1 + t_2 + t_3 + t_5 + t_6 = 6 + 6 + 4 + 4 + 4 + 4 = 28$. Third let $m = 7$ and $t_1 + t_2 + t_3 + t_5 + t_6 + t_7 = 4 + 4 + 4 + 4 + 4 + 4 + 4 = 28$. For $m = 5$, m is minimum number of colour classes. For $m = 6$, m neither minimum number of colour classes, nor maximum number of colour classes. For $m = 7$, m is maximum number of colour classes.

Example3.21. Let G be a graph of 13 cliques and 21 common vertices, each vertex is common between three cliques, with maximum clique number equal 5 and $\sum_{j=1}^{13} d(C_j) = 63$. First let $m = 6$ and $t_1 + t_2 + t_3 + t_5 + t_6 = 12 + 12 + 12 + 12 + 12 + 3 = 63$. Second let $m = 7$ and $t_1 + t_2 + t_3 + t_5 + t_6 + t_7 = 9 + 9 + 9 + 9 + 9 + 9 + 9 = 63$. For $m = 6$, m is minimum number of colour classes. For $m = 7$, m is maximum number of colour classes.

IV. METHOD OF FINDING MINIMUM NUMBER OF COLOUR CLASSES

This method is same as method of finding minimum number of colour classes for edge colouring in paper [11]. In this section we introduce the method as a result and prove the result, and introduce some results related to this method.

Proposition 4.1. Let G be a graph of k vertices $v_1, v_2, v_3, \ldots, v_k$ (common and non common) and $C_1, C_2, C_3, \ldots, C_w$ are proper cliques of these k vertices, with maximum cliques number equal n, we use the following method to partition these k vertices into m colour classes. In first step we try to find q_0 maximum number of colour classes each of common vertices only, and each class has appearance equal w at $C_1, C_2, C_3, \ldots, C_w$, such that it is impossible to be equal $q_0 + 1$. By impossible we mean there exist at least two common vertices belong to one colour class are adjacent. In second step we try to find q_1 maximum number of colour classes each of common vertices only, and each class has appearance equal $w-1$ at $C_1, C_2, C_3, \ldots, C_w$, such that it is impossible to be equal $q_1 + 1$. In third step we try to find q_2 maximum number of colour classes each of common vertices only, and each class has appearance equal $w-2$ at $C_1, C_2, C_3, \ldots, C_w$, such that it is impossible, to be equal $q_2 + 1$. In last step we try to find equal q_s maximum number of colour classes each of common vertices only, and each class has appearance equal $w-s$ at $C_1, C_2, C_3, \ldots, C_w$, such that it is impossible to be equal $q_s + 1$. After last step all common vertices partition to m colour classes. If $m = q_0 + q_1 + q_2 + \ldots + q_s$ then m is minimum number of colour classes.

Proof: Using Proposition 3.18. we partition common vertices, and neglect non common vertices.

Step I: Let $\sum_{j=1}^{w} d(C_j) = wq_0 + r$ where $0 \le r \le w-1$, using proposition 3.16. we have $\sum_{j=1}^{w} d(C_j) = wm \le wn$, or $\sum_{j=1}^{w} d(C_j) = w(m-1) + r \le wn$, suppose we need only step I, to partition these k sets into m colour classes, then $m \le n$, since it is a contradiction if $m < n$, then we have $m = n$ and m is minimum number of colour classes.

Step II: Let $\sum_{j=1}^{w} d(C_j) = wq_0 + (w-1)q_1 + r$ where $0 \le r \le w-2$, suppose we need only step I, and step II to partition these k sets into m colour classes, then we have $q_0 + q_1 + 1 = m$ if $r \ne 0$ and $q_0 + q_1 = m$ if $r = 0$, and suppose we partition these k sets into $m-1$ colour classes, then either there is q_0+x colour classes appear w times at $C_1, C_2, C_3, ..., C_w$ or there is q_1+x colour classes appear $w-1$ times at $C_1, C_2, C_3, ..., C_w$ where x is an integer $1 \le x$ which is a contradiction, therefore m is minimum number of colour classes.

Step III: Let $\sum_{j=1}^{w} d(v_j) = wq_0 + (w-1)q_1 + (w-2)q_2 + r$ where $0 \le r \le w-3$, suppose we need only step I, step II and step III to partition these k sets into m colour classes, then we have $\sum_{j=1}^{w} d(v_j) = wq_0 + (w-1)q_1 + (w-2)q_2 + r \le wn$ and $q_0 + q_1 + q_2 + 1 = m$ if $r \ne 0$ and $q_0 + q_1 + q_2 = m$ if $r = 0$, and suppose we partition these k vertices into $m-1$ colour classes, then either there is q_0+x colour classes appear w times a $C_1, C_2, C_3, ..., C_w$ or there is q_1+x colour classes appear $w-1$ times or there is q_2+x colour classes appear $w-2$ times at $C_1, C_2, C_3, ..., C_w$ where x is an integer $1 \le x$ which is a contradiction, because there is at least one colour class χ_i such that $w+1 \le t_i$, therefore m is minimum number of colour classes.

For the remaining steps we have $\sum_{j=1}^{w} d(C_j) = wq_0 + (w-1)q_1 + (w-2)q_2 + (w-3)q_3 + ... + (w-s)q_s + r$, in each following step, the proof is same as in step III. Therefore the result holds.

Remark 4.2. In this paper and coming papers we call the method in Proposition 4.1. method of finding minimum number of colour classes

Proposition4.3. Let G be a graph of k vertices $v_1, v_2, v_3, ..., v_k$ and $C_1, C_2, C_3, ..., C_w$ are proper cliques of these k vertices, with maximum clique number equal n, where $2 < n$, and there is a proper clique with clique number equal two. If all vertices (common vertices and non common vertices) partitioned into minimum m colour classes, then the number of colour classes appear w times at $C_1, C_2, C_3, ..., C_w$ not more than two.

Proof: Suppose the number of colour classes appear w times at $C_1, C_2, C_3, ..., C_w$ equal three or more than three, let χ_1, χ_2, χ_3 be three colour classes each appear w times at $C_1, C_2, C_3, ..., C_w$, from definition of colour class, each one of χ_1, χ_2, χ_3 appear at all of cliques $C_1, C_2, C_3, ..., C_w$, then there is no clique with clique number less than three, a contradiction since there is one clique with clique number equal two, it is also a contradiction for more than three colour classes each appears w times at $C_1, C_2, C_3, ..., C_w$, therefore the number of colour classes appear w times at $C_1, C_2, C_3, ..., C_w$, not more than two.

Next proposition is generalization of Proposition 4.3., the proof is same as Proposition 4.3.

Proposition 4.4. Suppose we have k vertices $v_1, v_2, v_3, ..., v_k$ and $C_1, C_2, C_3, ..., C_w$ all proper cliques of these k vertices, with maximum clique number equal n, and minimum clique number equal q, where $q < n$, all vertices (common vertices and non common vertices) partitioned into minimum m colour classes, then the number of colour classes appear w times at $C_1, C_2, C_3, ..., C_w$ not more than q.

Common vertices in the following proposition, not include quasi non common vertices, for more details you can see section five.

Proposition 4.5. Let G be a graph of k common vertices $v_1, v_2, v_3, ..., v_k$ and $C_1, C_2, C_3, ..., C_w$ are proper cliques of these k vertices, with maximum clique number equal n, let these k vertices be partitioned into m colour classes, where $m = n+1$, there is no colour class can appears W times, and there is n colour class appears

$w-1$ times at $C_1, C_2, C_3, ..., C_w$ and there is a colour class appears r times, where $2 \le r \le w-2$, if $n(w-1)+r=\sum_{j=1}^{w} d(C_j)$ then m is minimum number of colour classes.

Proof: To show m is minimum number of colour classes, suppose $m-1=n$, then there exists a colour class χ_i, where $1 \le i \le n$, and t_i be number of times the colour class χ_i appear at these w cliques, such that $t_i = w$ or $t_i = w+1$, a contradiction, therefore $m=n+1$ is minimum number of colour classes.

Corollary 4.6. Let G be a graph of k common vertices $v_1, v_2, v_3, ..., v_k$ and $C_1, C_2, C_3, ..., C_w$ are proper cliques of these k vertices, with maximum clique number equal n, let these k vertices be partitioned into m colour classes, where $m=n+1$, $k=k_1+k_2$, k_1 is number of common vertices and k_2 is number of non common vertices, there is n colour classes with maximum appearance of common vertices equal $w-1$ at $C_1, C_2, C_3, ..., C_w$ and there is a colour class with maximum appearance of common vertices equal r, where $2 \le r \le w-2$, if $n(w-1)+r=\sum_{l=1}^{k_1} a_l$, where $1 \le l \le k_1$, and a_l is appearance of the common vertex v_l. Let these k vertices be partitioned into m colour classes, where $m=n+1$, then m is minimum number of colour classes.

Proof: Since every non common vertices has appearance equal one at $C_1, C_2, C_3, ..., C_w$, then we can write $\sum_{i=1}^{k_2} b_i = k_2$, where b_i appearance of the common vertex v_i, $k_2+1 \le i \le k$, then we have $n(w-1)+r+k_2=\sum_{l=1}^{k_1} a_l + k_2 = \sum_{j+1}^{w} d(C_j) \le wn$. Using Proposition 3.18. addition of non common vertex does not changes number of colour classes for constant maximum clique number, therefore partition of k vertices into m colour classes, depend only on common vertices, therefore partition of k vertices into m colour classes, depend only on the equation $n(w-1)+r=\sum_{l=1}^{k_1} a_l$, using Proposition 4.5. we have m is minimum number of colour classes.

V. QUASI NONCOMMON VERTEX

In this section we introduce concept of quasi non common vertex between two cliques and between more than two cliques and introduce some results related to this concept.

Definition 5.1. Let G be a graph of common and non common vertices $v_1, v_2, v_3, ..., v_k$ (common between proper cliques $C_1, C_2, C_3, ..., C_w$), these k vertices partition into m minimum colour classes, a common vertex v_l, where $1 \le l \le k$, is called quasi non common vertex between two clique C_1 and C_2 if $d(C_1)+d(C_2)-1 \le m$.

Definition 5.2. Let G be a graph of common and non common vertices $v_1, v_2, v_3, ..., v_k$ (common between proper cliques $C_1, C_2, C_3, ..., C_w$), these k vertices partition into m colour classes, a common vertex v_l, where $1 \le l \le k$, is called quasi non common vertex between r cliques $C_1, C_2, C_3, ..., C_r$, if $\sum_{j=1}^{r} d(C_j) - r + 1 \le m.$

Proposition 5.3. Let G be a graph of k common and non common vertices $v_1, v_2, v_3, ..., v_k$ (common between proper cliques $C_1, C_2, C_3, ..., C_w$), if these k vertices partition into m colour classes, and there are two cliques C_i, C_j, and $d(C_i) + d(C_j) + 1 \le m$, if after addition v_{k+1} each of C_i, C_j include the vertex v_{k+1} and C_i becomes C_i^* and C_j becomes C_j^*, such that $d(C_i^*) = d(C_i) + 1$, and $d(C_j^*) = d(C_j) + 1$, then v_{k+1} is quasi non common vertex between C_i^* and C_j^*.

Proof: From $d(C_i^*) = d(C_i) + 1$, and $d(C_j^*) = d(C_j) + 1$, then inequality $d(C_i) + d(C_j) + 1 \le m$ can be written as $[d(C_i^*) - 1] + [d(C_j^*) - 1] + 1 \le m$, and $d(C_i^*) + d(C_j^*) - 1 \le m$, then v_{k+1} is quasi non common vertex between C_i^* and C_j^*.

Proposition 5.4. is generalization of Proposition 5.3. and proof of Proposition 5.4 is same as in Proposition 5.3.

Proposition 5.4. Let G be a graph of k common and non common vertices $v_1, v_2, v_3, ..., v_k$ (common between proper cliques $C_1, C_2, C_3, ..., C_w$), these k vertices partition into m colour classes, and there are r cliques $C_1, C_2, C_3, ..., C_r$, such that $\sum_{j=1}^{r} d(C_j) + 1 \le m$, if after addition v_{k+1} each of $C_1, C_2, C_3, ..., C_r$ include the vertex v_{k+1} and for all $1 \le j \le r$, C_j becomes C_j^*, such that $d(C_j^*) = d(C_j) + 1$, then v_{k+1} is quasi non common vertex between $C_1^*, C_2^*, C_3^*, ..., C^*_r$.

Proposition 5.5. Let G be a graph of k common and non common vertices $v_1, v_2, v_3, ..., v_k$ (common between proper cliques $C_1, C_2, C_3, ..., C_w$), if these k vertices partition into m colour classes and if we add only one quasi non common vertex v_{k+1} between two cliques, without changing maximum clique number n, then these $k+1$ vertices partitioned into m colour classes.

Proof: Let these m colour classes labelled by $1,2,3,...,m$, since after addition of a quasi non common vertex v_{k+1} the proper cliques remains $C_1, C_2, C_3, ..., C_w$, then there are two cliques C_i and C_j each of C_i, C_j include the vertex v_{k+1} where $1 \le i < j \le w$, and after addition of v_{k+1}, C_i becomes C_i^* and C_j becomes C_j^*, such that $d(C_i^*) = d(C_i) + 1$, and $d(C_j^*) = d(C_j) + 1$, let before addition $d(C_i) = p$, $d(C_j) = q$, and after addition, we have $d(C_i^*) = p + 1$, $d(C_j^*) = q + 1$, where $2 \le p < n$, $2 \le q < n$, let C_i labelled by $C_i \equiv (\alpha_1, \alpha_2, \alpha_3, ..., \alpha_p)$, and C_j labelled by $C_j \equiv (\beta_1, \beta_2, \beta_3, ..., \beta_q)$, where $\{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_p\} \subset \{1, 2, 3, ..., m\}$, $\{\beta_1, \beta_2, \beta_3, ..., \beta_q\} \subset \{1, 2, 3, ..., m\}$, then after addition of the quasi non common vertex v_{k+1} we labelled C_i^* by as $C_i^* \equiv (\alpha_1, \alpha_2, \alpha_3, ..., \alpha_p, \gamma)$, and labelled C_j^* by

$C_j^* \equiv (\beta_1, \beta_2, \beta_3, ..., \beta_q, \gamma)$, where $\{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_p, \gamma\} \subseteq \{1,2,3,...,m\}$, $\{\beta_1, \beta_2, \beta_3, ..., \beta_q, \gamma\} \subseteq \{1,2,3,...,m\}$, and $\gamma \le m$, $\gamma \notin \{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_p\}$, $\gamma \notin \{\beta_1, \beta_2, \beta_3, ..., \beta_q\}$, from the labels used, the vertex v_{k+1} labelled by γ, and v_{k+1} belongs to colour class χ_γ, where $1 \le \gamma \le m$. From Definition 3.21. we have $d(C_j^*) + d(C_l^*) - 1 = (p+1) + (q+1) - 1 = p + q + 1 \le m$, before we add v_{k+1} , we have $d(C_i) \le n-1$ and $d(C_j) \le n-1$ then v_{k+1} can belongs to one of the following colour classes $\chi_{p+1}, \chi_{p+2}, \chi_{p+3}, \cdots, \chi_m$ and $\chi_{q+1}, \chi_{q+2}, \chi_{q+3}, \cdots, \chi_m$, this means $\gamma \in \{p+1, p+2, p+3, ..., m\}$ and $\gamma \in \{q+1, q+2, q+3, ..., m\}$. Then these $k+1$ common and non common vertices partitioned into m colour classes.

Proposition 5.6. Let G be a graph of k common and non common vertices $v_1, v_2, v_3, ..., v_k$ (common between proper cliques $C_1, C_2, C_3, ..., C_w$), if these k vertices partition into m colour classes and if we add only one quasi non common vertex v_{k+1} between r cliques, without changing maximum clique number n, then these $k+1$ vertices partitioned into m colour classes.

Proof: Let these m colour classes labelled by $1,2,3,..., m$, since after addition of a quasi non common vertex v_{k+1} the proper cliques remains $C_1, C_2, C_3, ..., C_w$, then there are r cliques $C_{i1}, C_{i2}, C_{i2}, ..., C_{ir}$, where $r \le w$, and $\{C_{i1}, C_{i2}, C_{i2}, ..., C_{ir}\} \subseteq \{C_1, C_2, C_3, ..., C_w\}$, such that for all j, l, $1 \le j \le w$, $1 \le l \le r$, we have $C_{il} = C_j$, and after addition of v_{k+1}, C_{il} becomes C_{il}^*,

such that and $d(C_{il}^*) = d(C_j^*) = d(C_{il}) + 1 = d(C_j) + 1$, suppose before addition of vertex v_{k+1} we have $d(C_{il}) = p_l$, and after addition of vertex v_{k+1} we have $d(C_{il}^*) = p_l + 1$, where $2 \le p_l < n$, let C_{il} be labelled by $C_{il} \equiv (\alpha_{l1}, \alpha_{l2}, \alpha_{l3}, ..., \alpha_{lp})$, where $\{\alpha_{l1}, \alpha_{l2}, \alpha_{l3}, ..., \alpha_{lp}\} \subset \{1,2,3,...,m\}$, and C_{il}^* be labelled by $C_{il}^* \equiv (\alpha_{l1}, \alpha_{l2}, \alpha_{l3}, ..., \alpha_{lp}, \gamma)$, where $\{\alpha_{l1}, \alpha_{l2}, \alpha_{l3}, ..., \alpha_{lp}, \gamma\} \subseteq \{1,2,3,...,m\}$, and $\gamma \le m$, $\gamma \notin \{\alpha_{l1}, \alpha_{l2}, \alpha_{l3}, ..., \alpha_{lp}\}$, for all $1 \le l \le r$, the vertex v_{k+1} labelled by γ, and v_{k+1} belongs to colour class χ_γ, where $1 \le \gamma \le m$. From Definition 3.22. we have $\sum_{l=1}^{r} d(C_{il}^*) - r + 1 = \sum_{l=1}^{r} (p_{il} + 1) - r + 1 = \sum_{l=1}^{r} p_{il} + 1 \le m$. Before we add v_{k+1} , we have for all $1 \le l \le r$, $d(C_{il}) \le n-1$, then v_{k+1} can belongs to one of the following colour classes $\chi_{p_1+1}, \chi_{p_1+2}, \chi_{p_1+3}, \cdots, \chi_m$ and $\chi_{p_2+1}, \chi_{p_2+2}, \chi_{p_2+3}, \cdots, \chi_m$ and $\chi_{p_3+1}, \chi_{p_3+2}, \chi_{p_3+3}, \cdots, \chi_m$ and so on till $\chi_{p_r+1}, \chi_{p_r+2}, \chi_{p_r+3}, \cdots, \chi_m$, this means $\gamma \in \{p_1+1, p_1+2, p_1+3, ..., m\}$ and $\gamma \in \{p_2+1, p_2+2, p_2+3, ..., m\}$ and $\gamma \in \{p_3+1, p_3+2, p_3+3, ..., m\}$ and so on till $\gamma \in \{p_r+1, p_r+2, p_r+3, ..., m\}$. Then these $k+1$ common and non common vertices partitioned into m colour classes.

Corollary 5.7. Let G be a graph of k vertices $v_1, v_2, v_3, ..., v_k$ and $C_1, C_2, C_3, ..., C_w$ be proper cliques of these vertices, if these k vertices partition into m colour classes and if remove one quasi non common vertex between two cliques, without changing maximum clique number n, then these $k-1$vertices partitioned into m colour classes.

Proof: Let the vertices $v_1, v_2, v_3, \ldots, v_{k-1}$ be partitioned into m colour classes, if we add the vertex v_k between C_i and C_j without changing maximum clique number n, such that the vertex v_k be quasi non common vertex between C_i and C_j, where $1 \le i < j \le w$, using Proposition 5.5.then these k vertices partition into m colour classes, if we remove the vertex v_k, then these $k-1$vertices partitioned into m colour classes.

Proposition 5.8. Let G be a graph of k common and non common vertices $v_1, v_2, v_3, \ldots, v_k$ (each common vertex is common between two proper cliques of $C_1, C_2, C_3, \ldots, C_w$),where w odd, with maximum clique number equal n, if these k vertices partition into m colour classes, then $n \le m \le 2n-1$, if $m = 2n-1$ then we have $2n \le w+1$.

Proof: Using Proposition 3.16. we can write $\sum_{i=1}^{m} t_i \le wn$, to obtain maximum of colour classes we have minimum value of t_i. From definition of maximum clique number we have $n \le m$. If w odd then minimum value of t_i is $t_i = \frac{w+1}{2}$, for all $1 \le i \le m$, if $m \ge 2n$, then $(2n+l)\left(\frac{w+1}{2}\right) \le wn$, where $l = 0,1,2,3 \ldots$ then we have $2n + wl + l \le 0$, a contradiction, therefore $n \le m \le 2n-1$. If $m = 2n-1$ then $(2n-1)\left(\frac{w+1}{2}\right) \le wn$, then $2n \le w+1$.

Proposition 5.9. Let G be a graph of k vertices $v_1, v_2, v_3, \ldots, v_k$ (each common vertex is common between two proper cliques of $C_1, C_2, C_3, \ldots, C_w$), where w even, with maximum clique number equal n, if these k vertices partition into m colour classes, then $n \le m \le 2n-1$, if $m = 2n-1$ then we have $4n \le w+4$.

Proof: Using Proposition 3.16. we can write $\sum_{i=1}^{m} t_i \le wn$, to obtain maximum of colour classes we have minimum value of t_i. From definition of maximum clique number we have $n \le m$. If w even then minimum value of t_i is $t_i = \frac{w+2}{2}$, for all $1 \le i \le m-1$, and $t_m = \frac{w}{2}$, if $m \ge 2n$, then $(2n-1+l)\left(\frac{w+2}{2}\right) + \frac{w}{2} \le wn$, where $l = 0,1,2,3 \ldots$ then we have $4n + wl + 2l \le 2$, a contradiction since $2 \le n$, therefore $n \le m \le 2n-1$. If $m = 2n-1$ then $(2n-2)\left(\frac{w+2}{2}\right) + \frac{w}{2} \le wn$, therefore $4n \le w+4$.

Using Proposition 3.18. ,Corollary 3.19., Proposition 5.6. and Corollary 5.7. we can state the following remark.

Remark 5.10. Addition of non common vertex or quasi non common vertex or does not change number of colour classes if maximum clique number not changed by this addition, and removal of non common vertex or quasi non common vertex or does not change number of colour classes if maximum clique number not changed by this removal.

Conclusion:

For families of disjoint sets colouring technique the word vertex has three meanings we have to distinguish between them, common vertex between number of cliques, non common vertex and quasi non common vertex between number of cliques. Addition or removal of non common vertex and quasi non common vertex does not change number of colour classes if maximum clique number unchanged.

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