Extended $\Phi -$ Contraction Mapping And Study of Periodic Point in Symmetric Spaces

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Abstract:
The objective of this manuscript to prove the result on existence of fixed point and the periodic points in the setting of symmetric space under extended $\phi -$ contraction mapping. Our result is a generalization of the result of R.P.Pant. ["Extended Φ contraction mapping", The Journal of Analysis, (2024), 1-10] and D.W. Boyd, J.S. Wong, ["On nonlinear contractions", Proc. Amer. Math. Soc. 20, 458–464 (1969)].

Keywords: Continuous mappings, Fixed points, Periodic points, Contractive mappings, Symmetric space.

AMS Subject Classification: Primary: 47H10, Secondary 54H25.

1. INTRODUCTION

In fixed point theory, the objective is to find mathematically valid mappings in which at least one element does not change. Originating from Brouwer's [12] fixed point theorem in 1912, topological fixed point theory focuses on continuous transformations. Meanwhile, discrete fixed point theory, stemming from Tarski's theorem in 1955, concentrates on mappings in discrete spaces. Metric fixed point theory, while its foundational concepts predated, owes its practical and widespread application to the contributions of the Polish mathematician Stefan Banach [7].

Meir and Keeler [23] proved that if a self-mapping f of a complete metric space (X, d) satisfies the some condition then f has a unique fixed point. Boyd and Wong [11] generalized the Banach contraction principle in complete metric spaces. Matkowski [22] applied the well-known Banach Contraction principle and the generalization of Boyd and Wong [4] and some of the result of Browder and kirk [1, 21].

These results motivated various other results for ϕ-ontractions [17, 24, 28, 29, 30, 31, 32, 34, 35, 37, 38]. R. P. Pant [25] extended the Banach contraction principle and he gave an application of condition to determining the cardinality of the fixed point set of mappings after that R. P. Pant [26] gave the extended version of the concept of Φ-contraction mappings to introduce the condition that applies to contraction mappings as well as nonexpansive mappings. R. P. Pant and V. Rakocevic [27] introduce a new, weaker form of continuity that is both necessary and sufficient for fixed point existence.

The exploration of fixed points within contraction mappings in symmetric spaces began with Cicchese's work [13]. Wilson [36] subsequently pioneered these spaces by relaxing the requirement of the triangle inequality from the metric constraints. Currently, there's a substantial body of literature discussing fixed

point theory within symmetric spaces $[1-6, 13-19, 32, 33]$.
In this paper we study the extended φ contraction mapping and periodic point in symmetric spaces. Our results are more general than the result of Pant$[26]$results.

2. MATHEMATICAL PRELIMINARIES

Definition 2.1 [20]. If T is a self-mapping of a set X then a point x in X is called an eventually fixed point of T if there exists a natural number N such that

$$T^{n+1}(x) = T^{n}(\text{x}) \text{ for } n \geq N.$$

If $T(x) = x$ then x is called a fixed point of T. A point x in X is called a periodic point of period n if $T^n x = x$. The least positive integer n for which $T^n x = x$ is called the prime period of x.

Definition 2.2 [20]. The set $\{x \in X: Tx = x\}$ is called the fixed point set of the mapping $T: X \to X$.

Definition 2.3 [36]. Let X be a non-empty set. A symmetric on a set X is a real valued function $d: X \times X \to \mathbb{R}$ such that,

i. $d(x,y) \geq 0, \forall x, y \in X$,
ii. $d(x,y) = 0 \iff x = y$,
iii. $d(x, y = d(y,x)$.

Let d be a symmetric on a set X and for $\varepsilon > 0$and any $x \in X$, let $B(x,\varepsilon) = \{y \in X: d(x,y) < \varepsilon\}$.A topology $t(d)$on X is given by $U \in t(d)$if and only if for each$x \in U, B(x,\varepsilon) \subset U$ for some $\varepsilon > 0$.
A symmetric d is a semi-metric if for each $x \in X$and each $\varepsilon > 0$,$B(x,\varepsilon)$, is a neighborhood of x in the topology $t(d)$.There are several concepts of completeness in this setting. A sequence is a $d-$ Cauchy if it satisfies the usual metric condition.
Note: In symmetric space triangle inequality property of metric space relaxed. Therefore, it is general than metric space.

Definition 2.4 [36]. Let (X, d) be a symmetric (semi-metric) space.

i. (X, d) is S-complete if for every $d - cauchy$ sequence $\{x_n\}$ there exist x in X with
$$\lim_{n\to\infty} d(x_n, x) = 0.$$
ii. (X, d) is $d - cauchy$ complete if for every $d - cauchy$ sequence $\{x_n\}$ there exist x in X with $\lim_{n\to\infty} x_n = x$ with respect to $t(d)$.
iii. $S: X \to X$ is d-Continuous if $\lim_{n\to\infty} d(x_n, x) = 0$ implies
$$\lim_{n\to\infty} d(Sx_n, Sx) = 0.$$
iv. $S: X \to X$is $t(d)$ continuous if $\lim_{n\to\infty} x_n = x$ with respect to$t(d)$ implies $\lim_{n\to\infty} S(x_n) = Sx$ with respect to $t(d)$.

The following two axioms were given by Wilson [36].

Definition 2.5. Let (X, d) be a symmetric (semi-metric) space.

W1: Given$\{x_n\}, x, y \; in \; X$, $d(x_n, x) \to 0$ and $d(x_n, y) \to 0 \Rightarrow x = y$.

W2: Given$\{x_n\}$, $\{y_n\}$, and $x, y\ in\ X d(x_n, x) \to 0$ and

$$d(x_n, y_n) \to 0 \Rightarrow d(y_n, x) \to 0.$$

3. MAIN RESULT

Theorem 3.1 Let (X, ρ) be a Complete Symmetric space and $T: X \to X$ be such that for each x, y in X with $x \neq Tx\ or\ y \neq Ty$ we have

$$\rho(Tx, Ty) \leq \phi(\rho(x, y)), \ldots\ldots(i)$$

where $\phi: R^+ \cup \{0\} \to R^+ \cup \{0\}$ is such that $\phi(t) < t\ for\ t > 0$.

If T is upper semi continuous from the right or if ϕ is non-decreasing and $\lim_{n\to\infty} \phi^n(t) = 0, t > 0$, then T has a fixed point. T has a unique fixed point ⟺(i) is satisfied for each $x \neq y$ in X.

Proof. Let (X, ρ) be a complete Symmetric space. For proving the result using (i) when $x = Tx$ and$y = Ty$, $\rho(Tx, Ty) = \rho(x, y)$. We can say that T is continuous and $\rho(Tx, Ty) \leq \rho(x, y)$for each $x, y\ in\ X$.

Let y_0 be any point in X and $\{y_n\}$ be the sequence defined by $y_n = Ty_{n-1}$, that is,$y_n = T^n y_0$. If $y_{n+1} = y_n$ for some$n$, then y_n is a fixed point of T and the theorem holds. Therefore, assume that $y_{n+1} \neq y_n$ for each n $\geq$ 0. Given an integer $p \geq 1$, let $d_n = \rho(y_n, y_{n+p})$. Then using (i), for each $n \geq 1$and $p \geq 1$ we have,

$$d_n = \rho(y_n, y_{n+p}) = \rho(Ty_{n-1,} Ty_{n+p-1}).$$
$$\leq \phi(\rho(y_{n-1,} y_{n+p-1})) = \phi(d_{n-1})$$
$$\leq \phi^2\left(\rho(y_{n-2,} y_{n+p-2})\right) = \phi^2(d_{n-2}) \ldots\ldots \leq \phi^n\left(\rho(y_{0,} y_p)\right) = \phi^n(d_0)$$

Since $\{d_n\}$ is a strictly decreasing in $\mathbb{R}^+$, there exists $L \geq 0$such that

$$\lim_{n\to\infty} d_n = L = \lim_{n\to\infty} \phi(d_n) \ldots\ldots\ldots\ldots (ii).$$

Now assume that satisfies Matkowski condition [10], that is, ϕ is nondecreasing and $\lim_{n\to\infty} \phi^n(t) = 0$ for each $t > 0$.Then$\lim_{n\to\infty}(d_n) = \lim_{n\to\infty} \phi^n(d_0) = 0$.This implies that $\{y_n\}$ is a Cauchy sequence.

Next assume that T satisfies Boyd and Wond [4] condition, that is,$\phi$is upper semi continuous from the right.

If $L > 0$ then we get $\lim_{n\to\infty} \sup \phi(d_n) \leq \phi(L) < L$. Which contradicts (ii) since$d_n > L$. Hence $\lim_{n\to\infty} d_n = \lim_{n\to\infty} \rho(y_n, y_{n+p}) = 0$ and $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists z in X such that $\lim_{n\to\infty} y_n = z$ and $\lim_{n\to\infty} Ty_n = z$. Continuity of T implies $\lim_{n\to\infty} Ty_n = Tz$, that is, $z = Tz$ and z is a fixed point of T. Further, let u be any point in X. Then, since $T^n y_0 \neq T^{n-1} y_0$ for each n, using (i) we get

$$\rho(T^n u, T^n y_0) \leq \phi\left(\rho(T^{n-1}u, T^{n-1}y_0)\right) \leq \phi^2\left(\rho(T^{n-2}u, T^{n-2}y_0)\right) \leq \cdots \ldots \leq \phi^n\left(\rho(u, y_0)\right).$$

Our assumptions on ϕ imply that $\lim_{n\to\infty} \rho(T^n u, T^n y_0) = 0$, that is, $\lim_{n\to\infty} T^n u = z$.

Thus, if there exists a point y_0 such that $T^{n+1} y_0 \neq T^n y_0$ for each n, then for each u in X the sequence of iterates$\{T^n u\}$converges to z and z is the unique fixed point of T.

Therefore, $T^{n+1} y_0 \neq T^n y_0$, $n \geq 0$, for some y_0 implies uniqueness of the fixed point.

Now, assume that condition (i) is satisfied for all x, yin X. Then Tcan have only one fixed point. Conversely, suppose that T has a unique fixed point. Then for distinctx, y we have $x \neq fx$ or $y \neq fy$ which implies that condition (i) holds for each$x \neq y$. It may be noted that a mapping T satisfying this theorem cannot possess periodic points of prime period ≥ 2.

If possible, suppose Tsatisfies Theorem 3.1 and x is a periodic point of T with prime period 2, that is, $T^2x = x$ but$Tx \neq x$. Then using (i) we get

$$\rho(Tx, T^2x\ ) \leq \phi\big(\rho(x, Tx)\big) < \rho(x, Tx) = \rho(T^2x, Tx\ ),$$

is a contradiction. It follows similarly that T cannot have periodic points of prime period > 2.

This completes the proof of the theorem.

Example 3.2 Let $(X, \rho\ )$ be the Symmetric space. Where $X = [1, \infty)$ and $\rho(x, y) = (x - y)^2 \forall x, y \in X$. Let $T: X \to X$ be the signum function $Tx = sgnx$ defined as

$$T(x) = \begin{cases} -1, if\ x < 0, \\ 0, if\ x = 0 \\ 1, if\ x > 0. \end{cases}$$

.Then $Tx = 1$ for each x and T has a unique fixed point at 1. T satisfy condition (i) with $\phi(t) = \left({}^1\!/_2\right)t$. If $x \neq 1$ then $Tx = T^2x$ and x is an eventually fixed point.

Example 3.3 Let $(X, \rho\ )$ be the Symmetric space. Where $X = [-2, -1] \cup [1, 2]$and $\rho(x, y) = (x - y)^2 \forall x, y \in X$. Let $T: X \to X$ be defined by

$$Tx = sgn\ x.$$

Then Tsatisfies condition (i) with $\phi(t) = \frac{t}{2}$ for $t \leq 2$ and$\phi(t) = 2$ if $t > 2$ and has two fixed points 1and-1. Also, ϕ is nondecreasing and $\lim_{n\to\infty} \phi^n(t) = 0$ for each $t > 0$.

Example 3.4 Let $F = \{re^{i\theta}: 0 \leq \theta \leq 2\pi, r = 1, 3, 3^2, \ldots\ldots\}$ be the self-similar family of concentric circles, each lying within larger circles having radii in a geometric progression, in the $x - y\ plane$. Let X be the set of points of intersection of F with the N rays beginning at the origin and respectively making angles $0, \frac{2\pi}{N}, 2\left(\frac{2\pi}{N}\right), 3(2\pi/N), \ldots, (N-1)(2\pi/N)$ measured counter clockwise with the positive$x - axis$and let ρ be the symmetric metric $\rho(x, y) = (x - y)^2 \forall x, y \in X$.

Define $T: X \to X$ by

$$T\big(re^{i\theta}\big) = \lceil {}^r\!/_3 \rceil e^{i\theta}.$$

Where $\lceil x \rceil$ denotes the least integer not less than. Then T satisfies condition (i) with

$$\phi(t) = \left({}^1\!/_2\right)t$$

and has N fixed points $e^{io}, e^{i2\pi/N}, e^{i2(2\pi/N)}, e^{i3(2\pi/N)}, \ldots\ldots e^{i(N-1)(2\pi/N)}$.

If $N = 1$ then T is a Banach contraction mapping and has a unique fixed point $e^{io} = 1$ .The restriction of T on each of the N rays is a Banach contraction and, hence, a $\phi - contraction$.

Example 3.5 Let $X = \{z = re^{i\theta}: 0 \leq \theta \leq 2\pi, r = 1, 3, 3^2, \ldots\ldots\}$ be the self-similar family of Concentric circles, each lying within larger circles having radii in a geometric progression, in the $x - y\ plane$ and let ρ be the symmetric metric $\rho(x, y) = (x - y)^2 \forall x, y \in X$.

Define $T: X \to X$ by

$$T(z) = z/|z| = z/r.$$

Then T satisfies (i) with $\phi(t) = (1/2)t$ that is, T satisfies a Banach contraction condition if $x \neq Tx$ or $y \neq Ty$. Each point on the unit circle $z = e^{i\theta}$ is a fixed point while every other point is an eventually fixed point. In this example, the unit circle is a fixed circle.

Example 3.6 Let X be the subset of the xy −plane given by $X = \{z = re^{i\theta}: 0 \leq \theta \leq 2\pi, r = 1 \; or \; r \geq 3\}$ and ρ be the symmetric metric $\rho(x, y) = (x - y)^2 \forall x, y \in X$.
Define $T: X \to X$ by

$$T(z) = z/|z| = z/r.$$

Then T satisfies (i) with $\phi(t) = (1/2)t$ and each point on the unit circle $z = e^{i\theta}$ is a fixed point while every other point is an eventually fixed point satisfying $T^2x = Tx$. The unit circle is, thus, a fixed circle.

4. CONCLUSION

This research paper is the presentation of finding of fixed point and the periodic points in the setting of symmetric space under extended $\phi -$ contraction mapping. The results are the generalization of the result of R.P.Pant [26]. This paper is also generalized the work of Boyd and Wong and Matkowski. In the future this result can be obtained in the probabilistic metric space and fuzzy metric space.

Conflict of interest: We are declaring that this research paper has not been previously published and is not currently under consideration by another journal.

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