International Journal for Multidisciplinary Research (IJFMR)

E-ISSN: 2582-2160 ● Website: www.ijfmr.com ● Email: editor@ijfmr.com

Study of Fixed Point and Periodic Point Theorems in Symmetric Spaces

Arvind Bhatt[1], Radha[2]

[1,2]Department of Mathematics, School of science, Uttarakhand Open University, Haldwani, Uttarakhand, India, 263139

Abstract:
The objective of this manuscript to introduce a more relaxed form of continuity that serves as a necessary and sufficient condition for the existence of fixed points andthe mappings not only allow for fixed points but also admit periodic points, leading to fascinating combinations of fixed and periodic points in the setting of symmetric space.Our result is a generalization ofthe result of R.P.Pant et al[“Fixed point and periodic point theorems”, Acta ScientiarumMathematicarum, University of Szeged 2024.]

Keywords: Continuous mappings, Fixed points, Periodic points,Contractive mappings, Symmetric space, Asymptotic continuity.

AMS Subject Classification: Primary: 47H10, Secondary 54H25.

1. INTRODUCTION

Fixed point and periodic point theorems play a crucial role in various fields of mathematics, including analysis, topology, and applied mathematics. These theorems provide essential insights into the behavior of functions and mappings, offering a foundational framework for understanding various mathematical phenomena.Originating from Brouwer’s [12] fixed point theorem in 1912, topological fixed point theory focuses on continuous transformations. Meanwhile, discrete fixed point theory, stemming from Tarski’s theorem in 1955, concentrates on mappings in discrete spaces. Metric fixed point theory, while its foundational concepts predated, owes its practical and widespread application to the contributions of the Polish mathematician Stefan Banach [7].

Meir and Keeler [23] proved that if a self-mapping f of a complete metric space (X, d) satisfies the some condition then f has a unique fixed point. Rhoades, Park and Moon [6]gave a new result which encompasses most of such generalizations of the Meir-Keeler theorem.Boyd and Wong [11] generalized the Banach contraction principle in complete metric spaces.Jachymski [16] introduced a modified (ε, δ) condition to establish a unique common fixed point for two self-mappings $A$and Bon a complete metric space(X, d). Caristi’s [8]theorem contain fixed point theorem applied with the well-known Banach Contraction principle and the generalization of Kannan[34, 35], Chatterjee [25] and Ciric[26, 27].R. P. Pant and V. Rakocevic [29] introduce a new, weaker form of continuity that is both necessary and sufficient for fixed point existence.

The exploration of fixed points within contraction mappings in symmetric spaces began with Cicchese's work [13]. Wilson [38] subsequently pioneered these spaces by relaxing the requirement of the triangle

inequality from the metric constraints. Currently, there's a substantial body of literature discussing fixed point theory within symmetric spaces $[1-5, 13-19, 32, 33]$.
In this paper we study the fixed point and periodic point in symmetric spaces. Our findings are relevant to both contractive and non-expansive mappings. Moreover, our theorems stand independently of nearly all previously established results for contractive type mappings.Our results are more general than the result of Pant et al [29]results.

2. MATHEMATICAL PRELIMINARIES

Definition $\mathbf{2.1}$[$\mathbf{20}$]. If f is a self-mapping of a set X then a point x in X is called an eventually fixed point of $f$if there exists a natural number N such that
$f^{n+1}(x) = f^n(x)$ for$n \geq N$.
If $f(x) = x$ then x is called a fixed point of f. A point x in X is called a periodic point of period n if $f^n x = x$. The least positive integer n for which $\mathrm{f^n x = x}$ is called the prime period of x.The set of all iterates of a periodic point forms a periodic orbit which equals $\{x, fx, f^2x, \dots . f^{n-1}x\}$ if x is a periodic point of prime period n. A point x is called eventually periodic of period k if there exists $\mathrm{N} > 0$ such that $f^{n+k}(x) = f^n(x)$for $n \geq N$.

Definition $\mathbf{2.2}$ [$\mathbf{20}$]. The set $\{x \in X: Tx = x\}$ is called the fixed point set of the mapping $T: X \rightarrow X$.

Definition $\mathbf{2.3}$ [$\mathbf{38}$]. Let X be a non-empty set. A symmetric on a set X is a real valued function $d: X \times X \rightarrow \mathbb{R}$ such that,

i. $d(x, y) \geq 0, \forall x, y \in X$,
ii. $d(x, y) = 0 \iff x = y$,
iii. $d(x, y = d(y, x)$.

Let d be a symmetric on a set X and for $\varepsilon > 0$and any $x \in X$,let $B(x, \varepsilon) = \{y \in X: d(x, y) < \varepsilon\}$.A topology $t(d)$on X is given by $U \in t(d)$if and only if for each$x \in U, B(x, \varepsilon) \subset U$ for some $\varepsilon > 0$.
A symmetric d is a semi-metric if for each $x \in X$and each $\varepsilon > 0$,$B(x, \varepsilon)$, is a neighborhood of x in the topology $t(d)$.There are several concepts of completeness in this setting. A sequence is a $d-$ Cauchy if it satisfies the usual metric condition.

Definition $\mathbf{2.4}$[$\mathbf{38}$]. Let (X, d) be a symmetric (semi-metric) space.

i. (X, d) is S-complete if for every $d - cauchy$ sequence $\{x_n\}$ there exist x in X with $\lim\limits_{n \to \infty} d(x_n, x) = 0$.
ii. (X, d) is $d - cauchy$ complete if for every $d - cauchy$ sequence $\{x_n\}$ there exist x in X with $\lim\limits_{n \to \infty} x_n = x$ with respect to $t(d)$.
iii. $S: X \rightarrow X$ is d-Continuous if $\lim\limits_{n \to \infty} d(x_n, x) = 0$ implies $\lim\limits_{n \to \infty} d(Sx_n, Sx) = 0$.
iv. $S: X \rightarrow X$is $t(d)$ continuous if $\lim\limits_{n \to \infty} x_n = x$ with respect to$t(d)$ implies $\lim\limits_{n \to \infty} S(x_n) = Sx$ with respect to $t(d)$.

The following two axioms were given by Wilson [36].
Definition 2.5. Let (X,d) be a symmetric (semi-metric) space.

W1: Given $\{x_n\}$, $x, y\ in\ X$, $d(x_n,x) \to 0$ and $d(x_n,y) \to 0 \Rightarrow x = y$.
W2: Given $\{x_n\}$, $\{y_n\}$, and $x, y\ in\ X d(x_n,x) \to 0$ and
$d(x_n,y_n) \to 0 \Rightarrow d(y_n,x) \to 0$.

Definition 2.6 A function $f: X \to Y$ is called continuous if $\lim_{n\to\infty} fx_n = ft$ whenever $\{x_n\}$ a sequence in X such that $\lim_{n\to\infty} x_n = t$.

Definition 2.7.([41]) A self-mapping f of a metric space X is called $k - continuous, k = 1,2,3 \ldots..$ if $\lim_{n\to\infty} f^k x_n = ft$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} f^{k-1}x_n = t$.

Definition 2.8. A function $f: X \to Y$ will be called asymptotically k −continuous (or equivalently, asymptotically continuous) if $\lim_{k,n\to\infty} f(f^k x_n) = ft$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{k,n\to\infty} f^k x_n = t$.

Definition 2.9([26, 27]). If f is a self-mapping of a metric space (X,d) then the set $O(x,f) =$ is called the orbit of f at x and f is called orbitally continuous if $u = \lim_i f^{m_i} x$ implies $fu = \lim_i f f^{m_i} x$. Continuity implies orbital continuity but not conversely [26, 27].

Definition 2.10. ([42]).A self-mapping f of a metric space (X,d) is called weakly orbitally continuous if the set $\left\{y \in X: \lim_i f^{m_i} y = u \Rightarrow \lim_i f f^{m_i} y = fu\right\}$ is nonempty whenever the set $\left\{x \in X: \lim_i f^{m_i} x = u\right\}$ is nonempty.

Definition 2.11.([20]).Let f be a self- mapping of the set of real numbers and p be a periodic point of prime period n. The point p is called hyperbolic if, $|(f^n)'(p)| \neq 1$. The point p is called non-hyperbolic if, $|(f^n)'(p)| = 1$. Here $(f^n)'(p)$ denotes the derivative of $f^n(x)$ at p.

Definition 2.12. The function $f: (-\infty,\infty) \to (-\infty,\infty)$ such that $f(x)$ is the least integer not less than x is called the least integer function or the ceiling function and is denoted by $f(x) = \lceil x \rceil$.

3. MAIN RESULT

In the following theorems, we shall denote $\mathrm{m(x,y) = max\{d(x,fx), d(y,fy)\}}$.
Theorem 3.1. Let f be a self-mapping of a symmetric metric space (X,d). Suppose that given $\varepsilon > 0$ *there exist* $0 < \delta(\varepsilon) < \varepsilon$ such that for each $\mathrm{x, y}$ in X
(i) $\varepsilon < m(\mathrm{x,y}) \le \varepsilon + \delta \Longrightarrow \mathrm{d(fx,fy)} \le \varepsilon$, and
(ii) $\varepsilon - \delta < m(\mathrm{x,y}) \le \varepsilon \Longrightarrow \mathrm{d(fx,fy)} \le \varepsilon - \delta$.

Then either f has a unique fixed point and for each x in X the sequence of iterates $\{f^n x\}$converges to the fixed point, or each x in X is a fixed point or an eventually fixed point. The mapping f possesses a unique fixed point if and only if $x \neq y \Longrightarrow \max\{d(fx, fy)\} > 0$.

Proof. If each x in X is a fixed point of f then the conditions (i) and (ii) are satisfied since there exists no pair (x, y)in X that violates (i) and (ii). Thus, either each point of X is a fixed point of f or there exists a point x_0 in X such that $fx_0 \neq x_0$. Define a sequence $\{x_n\}$ in Xby $x_n = fx_{n-1}$, that is, $x_n = f^n x_0$. Condition (ii) implies that f cannot have periodic points of prime period ≥ 2. For example, suppose y is a periodic point of prime period $n \geq 2$, that is, $f^n y = y$ but $f^i y \neq f^n y$ for $1 \leq i < n$.Then $\max\{d(y, fy), d(fy, f^2 y), \dots . d(f^{n-1}y, f^n y)\} > 0$.
Let $\max\{d(y, fy), d(fy, f^2 y), \dots . d(f^{n-1}y, f^n y)\} = d(f^{k-1}y, f^k y) = \varepsilon > 0, 1 \leq k \leq n$. If $k = 1$ then $d(y, fy) = \varepsilon$ and $\max\{d(y, fy), d(f^{n-1}y, f^n y)\} = \varepsilon > 0$.This can also be written as $m(y, f^{n-1}y) = m(f^n y, f^{n-1}y) = \max\{d(f^{n-1}y, f^n y), d(f^n y, f^{n+1}y)\} = \varepsilon > 0$.
Using condition (ii) this yields $d(f^n y, f^{n+1}y) = d(y, fy) < \varepsilon$, a contradiction. Similarly, if $k > 1$ then $d\left(f^{k-1}y, f^k y\right) = \varepsilon$ and
$m\left(f^{k-2}y, f^{k-1}y\right) = \max\{d\left(f^{k-2}y, f^{k-1}y\right), d(f^{k-1}y, f^k y)\} = d(f^{k-1}y, f^k y)$
$= \varepsilon > 0$.
By virtue of (ii) this gives $d(f^{k-1}y, f^k y) < \varepsilon$, a contradiction. Hence, f cannot have periodic points of prime period ≥ 2 if f satisfies (ii). It may be observed that this conclusion remains true if we replace condition (ii) by any of the following two conditions:

$$\max\{d(x, fx), d(y, fy)\} = \varepsilon \Rightarrow d(fx, fy) < \varepsilon,$$

$\max\{d(x, fx), d(y, fy)\} = \varepsilon \Rightarrow d(fx, fy) > \varepsilon$.(3.1)
The above computation shows that fcan possess periodic points only if the set
$\left\{y \in X: \max\left\{d\left(f^{k-2}y, f^{k-1}y\right), d\left(f^{k-1}y, f^k y\right)\right\} = d(f^{k-1}y, f^k y > 0, k \geq 2\right\}$is nonempty, that is, condition (ii) will not hold in that case and some weaker condition, say,

$$d(fx, fy) \leq m(x, y), x, y \in X, \qquad (3.2)$$

will hold.
Also, if

$$\left\{y \in X: \max\left\{d\left(f^{k-2}y, f^{k-1}y\right), d\left(f^{k-1}y, f^k y\right)\right\} = d(f^{k-1}y, f^k y > 0, k \geq 2\right\} = X \qquad (3.3)$$

then each point of X will be a periodic point of period ≥ 2.
Therefore, condition (ii) implies that either $x_n = x_{n+1}$ for some $n \geq 1$ or $x_n \neq x_{n+1}$for each $n > 0$. If $x_n = x_{n+1}$for some $n \geq 1$ then $f^n x_0 = f^{n+1}x_0$, that is, x_n is a fixed point of f and x_0 is an eventually fixed point. On the other hand, if $x_n \neq x_{n+1}$for each n, then x_0 is neither a fixed point nor an eventually fixed point. This implies that either each point in X is a fixed point or an eventually fixed point, or there exists a point x_0 in X such that the sequence of iterates $\{f^n x_0\}$ consists of distinct points. Suppose the sequence of iterates $\{f^n x_0\}$of $x_0$consists of distinct points. Then $d\left(x_{n,} fx_{n,}\right) > 0, d(x_{n+1}, fx_{n+1}) > 0$ and there exists $\varepsilon > 0$ such that $\max\left\{d\left(x_{n,} fx_{n,}\right), d(x_{n+1}, fx_{n+1})\right\} = \varepsilon$.
By virtue of (ii) the last equation implies $d(fx_{n,} fx_{n+1}) \leq \varepsilon - \delta(\varepsilon)$, that is,
$d(x_{n+1}, x_{n+2}) \leq \varepsilon - \delta$. Thus, $d(x_{n+1}, x_{n+2}) = \varepsilon$and, $d(x_{n+1}, x_{n+2}) \leq \varepsilon - \delta$.Therefore, $d(x_n, x_{n+1})$ is a strictly decreasing sequence and, hence, tends to a limit $r \geq 0$. Suppose

$r > 0$ then there exists positive integer N such that

$$n \geq N \Rightarrow r < d(x_n, x_{n+1}) < r + \delta(r), 0 < \delta(r) < r. \tag{3.4}$$

This yields $r < \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = \max\{d(x_n, fx_n), d(x_{n+1}, fx_{n+1})\} < r + \delta(r)$, which by virtue of (i) yields $d(fx_n, fx_{n+1}) \leq r$, that is, $d(x_{n+1}, x_{n+2}) \leq r$. This contradicts (3.4). Hence $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Since $\{d(x_n, x_{n+1})\}$ is a strictly decreasing, for each $p \geq 1$ we get:

$$\max\left\{\{d(x_{n-1}, x_n), d(x_{n+p-1}, x_{n+p})\}\right\} = d(x_{n-1}, x_n) > 0.$$

Let $\max\{d(x_{n-1}, fx_{n-1}), d(x_{n+p-1}, fx_{n+p-1})\} = \varepsilon > 0$, that is, $d(x_{n-1}, x_n) = \varepsilon > 0$. Using (ii), we get $d(fx_{n-1}, fx_{n+p-1}) \leq \epsilon - \delta(\varepsilon)$, that is, $d(x_n, x_{n+p}) < d(x_{n-1}, x_n)$. Taking limit as $n \to \infty$ we get $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$.Hence$\{x_n\} = \{f^n x_0\}$ is a Cauchy sequence. Since X is complete, there exists z in X such that $x_n = f^n x_0 \to z \to z$. Also,$f^p x_n \to z$ for each $p > 0$ We claim that $z = fz$. If not, suppose $z \neq fz$ and $d(z, fz) = \varepsilon$. Then for sufficiently large values of n we get

$$m(x_n, z) = \max\{d(x_n, fx_n), d(z, fz)\} = d(z, fz) = \varepsilon. \tag{3.5}$$

Using condition (ii) the above inequality yields $d(fx_n, fz) \leq \varepsilon - \delta(\varepsilon)$. Making $n \to \infty$ we get $d(z, fz) \leq \varepsilon - \delta$, which contradicts (2.5). Hence $z = fz$ and z is a fixed point off.

Now, suppose that z and u are fixed points of f, that is, $z = fz$ and $u = fu$. Then $\max\{d(x_n, fx_n), d(u, fu)\} = d(x_n, fx_n) > 0$

By virtue of (ii), this implies $d(fx_n, fu) < d(x_n, fx_n)$. Taking limit as $n \to \infty$ we get $d(z, fu) = 0$, that is, $d(z, u) = 0$. Hence $z = u$ and z is the unique fixed point of f.

Moreover, if $y(\neq z)$ is any point in X then, by using (i) and (ii) it follows that $f^n y \to z$as $n \to \infty$.

The above analysis implies that if for some x_0 in X the sequence of iterates $\{f^n x_0\}$ consists of distinct points then f possesses a unique fixed point, say z, and $\lim_{n\to\infty} f^n y = z$ for each y in X. It further implies that if there does not exist an x_0 in X such that $\{f^n x_0\}$ consists of distinct points, then each point of X is either a fixed point or an eventually fixed point. Under the assumptions of this theorem, it is easy to verify that the condition:

$$x \neq y \Rightarrow \max\{d(x, fx), d(y, fy)\} > 0$$

is a necessary and sufficient condition for the existence of a unique fixed point. This proves the theorem.

Generalization Theorem 3.1 to include mappings that satisfy weaker conditions than (ii).

Example 3.2. Let (X, d)be a symmetric space and $d(x, y) = (x - y)^2 \ \forall x, y \in X$.

Let $f : X \to X$ be defined by $fx = x \ for \ each \ x \ in \ X$.Then for any $x, y \ in \ X$ we have $\max\{d(x, fx), d(y, fy)\} = \max\{d(x, x), d(y, y)\} = 0$.This implies that there exists no pair of points (x, y) in X which violates conditions (i) and (ii). Hence conditions (i) and (ii) of Theorem 3.1 are trivially satisfied and each x in X is a fixed point of f.

Theorem 3.3. Let f be a self-mapping of a symmetric metric space (X, d) such that

(iii) $d(fx, fy) < m(x, y)$, Whenever $m(x, y) > 0$,

(iv) Given$\varepsilon > 0$ there exists a $\delta > 0$ such that

$\varepsilon < m(x,y) \le \varepsilon + \delta \Rightarrow d(fx, fy) \le \varepsilon$.
Then f has a unique fixed point or each x in X is a fixed point or an eventually fixed point if and only if f is asymptotically k-continuous. If f has a unique fixed point, then for each x in X the sequence of iterates $\{f^n x\}$ converges to the fixed point.
Proof. If each x in X is a fixed point of f then the conditions (iii) and (iv) are satisfied trivially point of f or there exists some point which is not a fixed point. Suppose x_0 is not a fixed point of f, that is, $fx_0 \neq x_0$. Define a sequence $\{x_n\}$ in Xby $x_n = fx_{n-1}$, that is, $x_n = f^n x_0$. By virtue of (3.1), condition (iii) ensures that f does not possess periodic points of prime period ≥ 2. Therefore, either $x_n = x_{n+1}$ for some $n \geq 1$ or $\{x_n = f^n x_0\}$ consists of distinct points. If $x_n = x_{n+1}$ for some $n \geq 1$ then $x_n = f^n x_0$ is a fixed point and x_0 is an eventually fixed point. If $x_n \neq x_{n+1}$ for each n, then $d(x_n, fx_n) > 0$ for each n, and
$\max\{d(x_n, fx_n), d(x_{n+1}, fx_{n+1})\} > 0$.
Using (iii) this gives $d(fx_n, fx_{n+1}) < d(x_n, x_{n+1})$.Therefore, $\{d(x_n, x_{n+1})\}$ is a strictly decreasing sequence. Now, proceeding on the lines of the proof of Theorem 2.1 it follows that $\{x_n\} = \{f^n x_0\}$ is a Cauchy sequence. Since X is complete, there exists z in X such that $x_n = f^n x_0 \to z$. Also, $\lim_{n\to\infty} f^k x_n = z$ for each integer $k \geq 1$and $\lim_{k,n\to\infty} f^k x_n = z$. Moreover, using (iii), for each x in X we have $\lim_{n\to\infty} d(f^n x_0, f^n x) = 0$, that is, $\lim_{n\to\infty} f^n x = z$. Suppose that f is asymptotically continuous.
Then, since $\lim_{k,n\to\infty} (f^k x_n) = z$, asymptotic continuity implies $\lim_{k,n\to\infty} f(f^k x_n) = fz$.This implies $z = fz$ since $\lim_{k,n\to\infty} f(f^k x_n) = \lim_{k,n\to\infty} (f^{k+1} x_n) = z$. Therefore, z is a fixed point of f. If z and u are fixed points of f, then using (iii) it follows that $\lim_{n\to\infty} d(f^n x_0, f^n u) = 0$, that is, $d(z, u) = 0$. Hence, z is the unique fixed point of f.
If there does not exist an x_0 in X such that $x_n \neq x_{n+1}$ for each n, then each point of X is either a fixed point or an eventually fixed point. In this case the mapping f is obviously asymptotically continuous.
Conversely suppose that a mapping f satisfying (iii) and (iv) either has a unique fixed point or each x in X is a fixed point or an eventually fixed point. First suppose that f has a unique fixed point, say z. Then, by virtue of (iii) and (iv), for each x in X the sequence of iterates $\{f^n x\}$ is a Cauchy sequence and $\lim_{n\to\infty} f^n x = z = fz$. If $\{x_n\}$ is any sequence in X then for each n we have $\lim_{k\to\infty} f^k x_n = z$and $\lim_{k,n\to\infty} f(f^k x_n) = z = fz$. This implies that f is asymptotically continuous. On the other hand, if each point of X is either a fixed point or an eventually fixed point then f is obviously asymptotically continuous. This completes the proof of the theorem.

Example3.4. Let(X, d) be the Symmetric space.Where$X = [-\infty, \infty)$and $d(x,y) = (x-y)^2 \forall x, y \in X$. Define $f: X \to X$by $fx = 0$, for each x in X. Then f satisfies the conditions of Theorems 3.1and 3.3, possesses a unique fixed point $x = 0$ and every nonzero x is an eventually fixed point since $fx = f^2 x$ if $x \neq 0$.

Example 3.5. Let $X = \{\pm 4^n : n = 0, 1, 2, 3, \dots.\}$ and Let (X, d) be the Symmetric space.Where$X = [1, \infty)$and $d(x,y) = (x-y)^2\ \forall\, x, y \in X$. Let $f: X \to X$ be the signum function $fx = sgnx$ defined as $fx = 1\ if\ x > 0, f0 = 0, fx = -1\ if\ x < 0$Then$f$ is a continuous mapping which satisfies all the conditions of Theorems 3.1 and 3.3, has two fixed points $x = \pm 1$ and all other points are eventually fixed.

Examples 3.2 and 3.5 show that a mapping satisfying the conditions of theorems 3.1and theorem 3.3 may not satisfy any contractive condition.

Example 3.6. Let $X = \{re^{i\theta}: 0 \leq \theta \leq 2\pi, r = 1, 4, 4^2, \ldots..\}$be the self-similar family of circles, each lying within larger circles having radii in geometric progression, in the$x - y\, plane$ and let d be the symmetric metric $d(x,y) = (x-y)^2\ \forall\, x,y \in X$. Define $f: X \to X$ by $f(re^{i\theta}) = \lceil r/4 \rceil e^{i\theta}$. Where$[x]$ denotes the least integer not less than. Then f satisfies is a continuous mapping which satisfies all the conditions of Theorem 3.1 and has the points on the unit circle as fixed points, that is, the unit circle is a fixed circle of f.

Example3.7. Let $X = [0,2]$ and d be the Symmetric metric on X, and$d(x,y) = (x-y)^2 \forall x,y \in X$. Let $f: X \to X$ by $fx = 1\ if\ x \leq 1,\ fx = 0\ if\ x > 1,$ Then f satisfies the conditions of the Theorem 3.1, has a unique fixed point $x = 1,$ every other point is an eventually fixed point since $f^2x = f^3x$ for each $x \neq 1$, and for each x in X the sequence of iterates $\{f^n x\}$obviously converges to the fixed point as $n \to \infty$. The mapping f satisfies conditions (i) and (ii) with $\delta(\varepsilon) = (\varepsilon - 1)/2$ if $\varepsilon > 1, \delta(1) = 1/2,$ $\delta(\varepsilon) = (1-\varepsilon)/2$ if $1/2\ \leq \varepsilon < 1$and $\delta(\varepsilon) = \varepsilon/2$ if $\varepsilon < 1/2$ and satisfies the contractive condition $d(fx,fy) < max\{d(x,fx),\ d(y,fy)\}$whenever$max\{d(x,fx),\ d(y,fy)\} > 0$. The mapping $f$is discontinuous at the fixed point and, therefore, the above theorem contains solutions to Rhoades problem [37] on continuity of contractive mappings at the fixed point. Also, f is a $2-continuous$ mapping and, hence, an asymptotically continuous mapping that satisfies the conditions of Theorem 3.3.

Example 3.8. Let $X = [0,\infty)$ and let d be the symmetric metric$d(x,y) = (x-y)^2\ \forall\, x,y \in X$. Define $f: X \to X$ by $fx = 1$if$x \leq 1,\ fx = \frac{x}{3}\ if\ x > 1.$ $f$satisfies the conditions of Theorem 3.3 and has a unique fixed point $x = 1$ at which f is discontinuous. This example provides a new solution of the Rhoades problem [34] in the form of an asymptotically continuous mapping. It is easy to see that each $y \neq 1$ is an eventually fixed point and $f^N y = f^{N+1}y = 1$for some integer $N = N(y) \geq 1$. The mapping f satisfies the (ε,δ) condition with $\delta(\varepsilon) = \frac{2}{3} - \varepsilon$ if $\varepsilon < 2/3$, $\delta(\varepsilon)=2-\varepsilon$ if $2/3 \leq \varepsilon < 2$ and $\delta(\varepsilon) = \varepsilon$ if $\varepsilon \geq 2$.

Example 3.9. Let $X = [0,2]$ and d be the symmetric metric and $d(x,y) = (x-y)^2\ \forall x,y \in X$. Let $f: X \to X$ by $fx = (2+x)/3$if $x < 1$, $fx = 0$if $x \geq 1$.Then f is not asymptotically continuous, satisfies the conditions (iii) and (iv) of Theorem (3.3) and does not have a fixed point. fSatisfies condition (iv) with $\delta(\varepsilon) = \min(\varepsilon, 1-\varepsilon)\ if\ \varepsilon < 1\delta(\varepsilon)$ and$\delta(\varepsilon) = \varepsilon$when$\varepsilon \geq 1$.

Example 3.10. Let $X = [0,2]$ equipped with the symmetric metric $d(x,y) = (x-y)^2\ \forall x,y \in X$. Let $f: X \to X$by $fx = x - [x]$, $[x]$ being the greatest integer $\leq x$ Then f is a $2-$continuous mapping that satisfies the conditions of Theorem 3.3. Each x in the interval [0,1) is a fixed point while each y in [1,2] is an eventually fixed point since $f^2y = fy$.

Theorem 3.11. (The (ε,δ) Fixed and Periodic Points Theorem). Let f be a k-continuous or asymptotically continuous self-mapping of a symmetric space (X,d) such that

(v) $max\{d(x,fx),\ d(y,fy)\} > 0 \Rightarrow d(fx,fy) \leq max\{d(x,fx), d(y,fy)\}$strict inequality holding if x,y are non-periodic,

(vi) Given$\varepsilon\ >\ 0$ there exists a $\delta\ >\ 0$ such that

$\varepsilon < \max\{d(x, fx), d(y, fy)\} \leq \varepsilon + \delta \Rightarrow d(fx, fy) \leq \varepsilon.$
Then either (a) each x in X is a fixed point or an eventually fixed point, or (b) if there exists an x_0 in X such that the sequence iterates $\{f^n x_0\}$ consists of distinct points then f possesses a unique fixed point, say z, and $\lim_{n\to\infty} f^n y = z$ provided $(y \neq z)$ is not a periodic point, or (c) if for each x in X, the sequence of iterates consists of only finitely many distinct points then f may possess periodic points besides fixed points or (d) each point is a periodic point of prime period > 1.

Proof. We first note that if each x in X is a fixed point of f then the conditions (v) and (vi) are satisfied trivially since there exists no pair (x, y) that violates (v) and (vi). Thus, either each x in X is a fixed point of f or there exists some point which is not a fixed point. Suppose x_0 is not a fixed point, that is, $fx_0 \neq x_0$. Define a sequence$\{x_n\}$ in X recursively by $x_n = fx_{n-1}$, that is,$x_n = f^n x_0$. Then two cases arise: either there exists $n > 1$ such that $x_n = x_i$ for some $i < n$ or for each n we have $x_n \neq x_i$. We first consider the case $x_n \neq x_i$for $i < n$, $n > 1$. In this case, x_0 is neither an eventually fixed point nor a periodic or eventually periodic point, that is, the sequence$\{x_n = f^n x_0\}$ consists of distinct points. Since $\{f^n x_0\}$ consists of distinct points, using (v) we get $d(f^n x_0, f^{n+1} x_0) < max\{d(f^{n-1} x_0, f^n x_0), d(f^n x_0, f^{n+1} x_0)\}$. This implies that $\{d(f^n x_0, f^{n+1} x_0)\} = \{d(x_n, x_{n+1})\}$is a strictly decreasing sequence and, hence, converges to a limit $r \geq 0$. Suppose $r > 0$ then there exists a positive integer N such that

$$n \geq N \Rightarrow r < \{d(x_n, x_{n+1})\} < r + \delta(r).$$

This yields $r < max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = \max\{d(x_n, fx_n), d(x_{n+1}, fx_{n+1})\} < r + \delta(r)$. By virtue of (vi) this gives $d(fx_n, fx_{n+1}) = d(x_{n+1}, x_{n+2}) \leq r$, a contradiction. Hence $r = 0$ and $d(x_n, x_{n+1}) \to 0$as $n \to \infty$. From this and (v) it follows easily that $\{x_n\} = \{f^n x_0\}$ is a Cauchy sequence. SinceX is symmetric, there exists z in X such that $x_n = f^n x_0 \to z$. Also, $\lim_{n\to\infty} f^k x_n = z$ for each integer $k \geq 1$and $\lim_{k,n\to\infty} (f^k x_n) = z$. Suppose f is asymptotically continuous. Since $\lim_{k,n\to\infty} (f^k x_n) = z$, asymptotic continuity of f implies $\lim_{k,n\to\infty} f(f^k x_n) = fz$. This implies $z = fz$since $\lim_{k,n\to\infty} f(f^k x_n) = \lim_{k,n\to\infty} (f^{k+1} x_n) = z$. Therefore, z is a fixed point of f. Now, suppose that z and u are fixed points of f, that is, $z = fz$ and $u = fu$. Then $\max\{d(x_n, fx_n), d(u, fu\} = d(x_n, fx_n) > 0$. By virtue of (v), this implies $d(fx_n, fu) < d(x_n, fx_n)$.Taking limit as $n \to \infty$ we get $d(z, fu) = 0$, that is, $z = u$ and z is the unique fixed point of f. This implies that if $y(\neq z)$ is not a periodic point then $f^m y = z$ for some $m > 1$ or $\{f^n y\}$ consists of distinct points. If $\{f^n y\}$ consists of distinct points then using (v) and (vi) it follows that$\{f^n y\}$ is a Cauchy sequence and $f^n y \to z$as $n \to \infty$. Thus, if there exists an x_0 in X such that the sequence iterates $\{f^n x_0\}$ consists of distinct points then f possesses a unique fixed point z and $\lim_{n\to\infty} f^n y = z$ for each non periodic $y(\neq z)$ point .This further implies that if $y(\neq z)$ does not satisfy $\lim_{n\to\infty} f^n y = z$then y is either a periodic point of prime period > 1 or $f^m y = z$ for some integer $m \geq 1$. Now let us consider the case when, for each xin X, the sequence of iterates consists of only finitely many distinct points. In this case if each point in X is not a fixed point or an eventually fixed point then f possesses periodic points besides fixed points and eventually fixed points. This implies that if f does not possess a fixed point then each point is a periodic point (e.g. Example 3.12 below). If f is $k-$ continuous then the theorem holds since $k-$ continuity implies asymptotic continuity. It is clear from the proof of this theorem that a mapping f satisfying the conditions of Theorem 3.12 is not a chaotic mapping since the sequence of iterates for each x in X either converges to a fixed point or forms a periodic orbit.

Example3.12. (Example 2.10 of [41]). Let $X = [-1,1]$ and d be the symmetric metric $d(x,y) = (x-y)^2 \ \forall x,y \in X$. Let $f: X \to X$by $fx = -|x|x$, that is, $fx = -x^2$ if $x \geq 0$ and $fx = x^2$ if $x < 0$. Then f satisfies condition (vi) of Theorem 3.11 with $\delta(\varepsilon) = \left(\sqrt{(\varepsilon/2)}\right) - (\varepsilon/2)$ if $\varepsilon < 2$ and $\delta(\varepsilon) = \varepsilon$if $\varepsilon \geq 2$, and also satisfies the condition (v). If we take $x = 1$ and$y = -1$then $fx = y$, $f^2x = x$, $fy = x$, $f^2y = y$ and $max\{d(x,fx), d(y,fy)\} = d(fx,fy) = 2$ and $(1,-1)$ is the only pair of elements in X satisfying such a condition, and all other pairs of points (x,y)satisfy the condition $d(fx,fy) < max\{d(x,fx), d(y,fy)\}$. Thus, as observed in the proof of Theorem 2.14, f has a unique fixed point $x = 0$, two periodic points$y = \pm 1$ and for each $y \neq \pm 1$ we have $\lim_{n\to\infty} f^n y = 0$. This also shows that f does not have eventually fixed points.

Example3.13. Let $X = [-1,1]$ and d be the symmetric metric $d(x,y) = (x-y)^2 \ \forall x,y \in X$. Let $f: X \to X fx = -[|x|]x$ where $[x]$denotes the greatest integer not greater than x. Then $fx = 0$if$-1 < x < 1$, $f1 = -1$, $f(-1) = 1$. As in Example 3.12 it is easy to show that $f$satisfies the conditions of Theorem 3.11, has a unique fixed point $x = 0$and two periodic points $y = \pm 1$ while every other point is an eventually fixed point.

Example 3.14. Let A, B, C be the vertices of an equilateral triangle with each side of length 1. Let D be the circumcentre of the triangle ABCandE, F be points on the perpendicular line to the plane of ABC through D such that $DE = DF = 1/2$. Let $X = \{A, B, C, D, E, F\}$ and d be the Symmetric metric on$X$and$d(x,y) = (x-y)^2 \ \forall x,y \in X$. Let $f: X \to X$ by $fA = B, fB = C, fC = A, fD = D, fE = F, fF = E$ Then f satisfies all the conditions of Theorem 3.11and has a unique fixed point D, two periodic points $E, F$of prime period 2 and three periodic points A, B, C of prime period 3.

4. CONCLUSION

The study of fixed points and periodic points is a vibrant and continually evolving area of mathematics.This research paper is the presentation of finding of fixed point and the periodic points in the setting of symmetric space.The results are the generalization of the result of R.P.Pant and Vladimir Rakocevic [29].

Conflict of interest: We are declaring that this research paper has not been previously published and is not currently under consideration by another journal.

REFERENCES

1. M. Aamri, D.El.,Moutawakil, Common fixed points for weakly compatible maps in symmetric spaces with application to probabilistic spaces, Appl. Math. E-Notes, 5(2005), 171–175.
2. M. Aamri, D.El.,Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl, 270(2002), 181-188.
3. M. Aamri, D.El.,Moutawakil, Common fixed points under contractive conditions in symmetric spaces, Appl. Math. E-Notes, 3(2003), 159–162.

4. M. Abbas, B.E. Rhoades, Common fixed point theorems for occasionally weakly compatible mappings satisfying a generalized contractive condition, Mathematical Communications, 13(2008), 295-301.
5. M. Abbas, B.E. Rhoades, Fixed and periodic point results in cone metric spaces, Applied Mathematics Letters, 22(4)(2009), 511-515.
6. Rhoades, B.E., Park, S., Moon, K.B.: On generalizations of the Meir-Keeler type contraction maps. J. Math. Anal. Appl. 146, 482–494 (1990).
7. S. Banach, Sur les operations dans les ensembles abstraits etleur application aux equations integrals, Fund. Math., 3(1922), 133-181.
8. Caristi, J.: Fixed point theorems for mappings satisfying inwardness conditions. Trans. Amer. Math. Soc. 215, 241–251 (1976)
9. A. Bhatt, H. Chandra, Occasionally weakly compatible mapping in cone metric space, Applied Mathematical Sciences, 6(55) (2012), 2711-2717.
10. R.K. Bisht, V. Rakočević, Generalized Meir-Keeler type contractions and discontinuity at fixed point, Fixed Point Theory, 19(1) (2018), 57-64.
11. D.W. Boyd, J.S. Wong, on nonlinear contractions, Proc. Amer. Math. Soc. 20 (1969), 458–464.
12. L. E. J. Brouwer, Uber Abidingness von Manning fastigiated, Math. Ann.71, (1912), 97–115.
13. M.Cicchese, Questioni di completezza e contrazioni in spazimetricigeneralizzati, Boll Un Mat Ital, 5: 175–179.
14. H. Chandra, A. Bhatt, Some fixed point theorems for set valued maps in symmetric spaces, International Journal of Mathematical Analysis, 3(17)(2009), 839-846.
15. Troy L. Hicks, B.E. Rhoades, Fixed point theory in symmetric spaces with applications to probabilistic spaces, Nonlinear Analysis: Theory, Methods & Applications 36(3) (1999), 331-344.
16. J. Jachymski, Common fixed point theorems for some families of maps, Indian J. Pure Appl. Math., 25 (1994), 925-34.
17. J. Jachymski, J. Matkowski, T. Świątkowski, Nonlinear contractions on semimetric spaces, J. Appl. Anal, 3(1) (1995), 125–134.
18. G. Jungck, B.E.Rhoades, Fixed point theorems for occasionally weakly compatible mappings, Fixed Point Theory, 7(2006), 286-296.
19. G. Jungck, B.E.Rhoades, Fixed point theorems for occasionally weakly compatible mappings, Erratum, Fixed Point Theory, 9(2008),383-384.
20. R.A. Holmgren, A first course in discrete dynamical systems, Springer-Verlag, New York, 1994.
21. W.A. Kirk, Contraction mappings and extensions, in Handbook of Metric Fixed Point Theory, Kluwer Academic Publishers, Dordrecht,The Netherlands, (2001) ,1–34.
22. J. Matkowski, Integrable solutions of functional equations,Dissertations Math, (1975), 127:68.
23. A. Meir, E. Keeler, A theorem on contraction mappings, J. Math. Anal. Appl. 28(1969), 326–329.
24. O. Nica, Existence results for second order three-point boundary value problems, Differ. Equ. Appl., 4(4) (2012), 547-70.
25. Chatterjea, S.K.: Fixed point theorems. C. R. Acad. Bulgare Sci. 25, 727–730 (1972)
26. Ciri´ ´ c, Lj.: On contraction type mappings. Math. Balkanica 1, 52–57 (1971)
27. Ciri´ ´ c, Lj.:Generalised contractions and fixed point theorems. Publ. Inst. Math. (Beograd) (NS) 25, 19–26 (1971)
28. R.P. Pant, Extended Φ contraction mapping, The Journal of Analysis, (2024), 1-10.

29. R.P. Pant, V. Rakočevič, Fixed point and periodic point theorems, ActaScientiarumMathematicarum, (2024),1-18.
30. R.P. Pant, A common fixed point theorem under a new condition, Indian J. Pure Appl. Math.30-2(1999), 147-152.
31. R.P. Pant, Common fixed point theorems for contractive maps, J Math. Anal. Appl. 226 (1998), 251-258.
32. R.P. Pant, N. Özgür, N. Taş, A. Pant, M.C.Joshi, New results on discontinuity at fixed points, J. Fixed Point Theory Appl. 22(2020), 1-14.
33. R.P. Pant, Common fixed points of noncommuting mappings, J. Math. Anal. Appl. 188-2(1994), 436-440
34. Kannan, R.: Some results on fixed points. Bull. Calcutta Math. Soc. 60, 71–76 (1968).
35. Kannan, R.: Some results on fixed points-II. Amer. Math. Monthly 76, 405–408 (1969) .
36. S.Reich, Fixed point of contractive functions. Boll. UMI, 5(1972), 26-42.
37. B.E.Rhoades, Contractive definitions and continuity, Contempory Math. 72(1988), 233-245.
38. W.A. Wilson, On semi-metric spaces, Amer. J. Math, 53(1931), 361–373.
39. X. Zhang, Common fixed point theorems for some new generalized contractive type mappings, J. Math. Anal. Appl., 333,(2007), 780-786.
40. Zheng, D. and Wang, P., “Weak $\theta - \varphi$ contractions and discontinuity”, J. Nonlinear Sci. Appl., 10(2017), 2318-2323.
41. Pant, A., Pant, R.P.: Fixed points and continuity of contractive maps. Filomat 31(11), 3501–3506 (2017)
42. Pant, Abhijit, Pant, R.P., Joshi, M.C.: Caristi type and Meir-Keeler type fixed point theorems. Filomat 33(12), 3711–3721 (2019)
43. Bisht, R.K., Rakoˇceviˇc, V.: Generalized Meir-Keeler type contractions and discontinuity at fixed point. Fixed Point Theory 19(1), 57–64 (2018)
44. Bisht, R.K.: Rakoˇceviˇc, fixed points of convex and generalized convex contractions. Rend. Circ. Mat. Palermo Ser. 2, 21–28 (2018)
45. Celik, U., Ozgur, N.: A new solution to the discontinuity problem on metric spaces. Turkish J. Math. 44(4), 1115–1126 (2020)
46. Devaney, R.L.: An introduction to chaotic dynamical systems. Benjamin/Cummings Publishing Co., California (1986)
47. Ozgur, N.Y., Tas, N.: Some fixed-circle theorems on metric spaces. Bull. Malays.Math. Sci. Soc. 42(4), 1433–1449 (2019)
48. Ozgur, N.: Fixed-disc results via simulation functions. Turkish J. Math. 43(6), 2794–2805 (2019)
49. Ozgur, N.Y., Tas, N.: Fixed-circle problem on S-metric spaces with a geometric viewpoint. FactaUniversitatis. Series: Math. Inf. 34(3), 459–472 (2019)
50. Ozgur, N.Y., Tas, N.: Some fixed-circle theorems and discontinuity at fixed circle. AIP Conf. Proc. 1926, 020048 (2018)
51. Pant, R.P.: Common fixed points of four mappings. Bull. Cal. Math. Soc. 90, 281–286 (1998)
52. Pant, R.P.: A Common fixed point theorem under a new condition. Indian J. Pure Appl. Math. 30(2), 147–152 (1999)
53. Pant, Abhijit, Pant, R.P., Joshi, M.C.: Caristi type and Meir-Keeler type fixed point theorems. Filomat 33(12), 3711–3721 (2019)

54. Pant, R.P., Ozgur, N.Y., Tas, N.: Discontinuity at fixed points with applications. Bull. Belgian Math. Soc. Simon Stevin 26–4, 571–589 (2019)
55. Rashid, M., Batool, I., Mehmood, N.: Discontinuous mappings at their fixed points and common fixed points with applications. J. Math. Anal. 9–1, 90–104 (2018)
56. Rhoades, B.E.: Contractive definitions and continuity. Contemp. Math. (Amer. Math. Soc.) 72, 233–245 (1988)
57. Saleh, H.N., Sessa, S., Alfaqih, W.M., Imdad, M., Mlaiki, N.: Fixed circle and fixed disc results for new types of Θc-contractive mappings in metric spaces. Symmetry 12(11), 1825 (2020)
58. Suzuki, T.: A generalized Banach contraction principle that characterizes metric Completeness. Proc. Amer. Math. Soc. 136–5, 1861–1869 (2008)
59. Wardowski, D.: Solving existence problems via F-contractions. Proc. Amer. Math. Soc. (2017).
60. Pant, A., Pant, R.P., Prakash, K.: Dynamics of a family of orbitally continuous mappings. Filomat 31(11), 3507–3517 (2017)