Nonlinear Functional Differential Equation in Banach Space: A Study of Its Solution

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Abstract:
Existence theorem for nonlinear functional differential equation in Banach space under the Lipschitz and Caratheodory conditions have been studied. The evidence of the existence theorem has been illustrated.

Keywords: Nonlinear functional differential equation, existence theorem, solution, Lipschits and Caratheodory conditions, hybrid fixed point theorem.

1. Introduction

Let $I = [-a, 0\,]$ and $I_+ = [0, a]$ be closed and bounded intervals in real line $\mathbb{R}$. Let $J = I \cup I_+$ is bounded and closed intervals in $\mathbb{R}$. The Banach space C of all continuous real-valued functions ϕ on I with the supremum norm $\|.\|_{\mathbb{C}}$ and multiplication '.' respectively, defined by

$$\|\phi\|_C = \sup_{t\in I}|\phi(t)| \quad \text{and} \quad (x.y)(t) = x(t).y(t), \ \ t \in I$$

Let's consider the first order nonlinear functional differential equation (FONFDE (1.1))

$$\frac{d}{dt}\left(\frac{x(t)}{f(t,x(t))}\right) = g\big(t, x_t, x(t)\big)\ , \quad t \in I_+ \quad \text{and} \quad x(t) = \phi(t)\ , \ t \in I \qquad (1.1)$$

Where, continuous function $f: I_+ \times\to \mathbb{R} - \{0\}$ and $g: I_+ \times C \times \mathbb{R} \to \mathbb{R}$, defined function $x_t(\alpha): I \to C$ by $x_t(\alpha) = x(t+\alpha)$ for all $\alpha \in I$.

Let $x \in C(J,\mathbb{R}) \cap AC(J,\mathbb{R}) \cap C(I,\mathbb{R})$ be the solution of FONFDE (1.1), where $AC(J,\mathbb{R})$ indicates the space of all absolutely continuous real-valued functions from $J\ to\ \mathbb{R}$.
Since a very long back the active area of research is the nonlinear functional differential equations [13-14]. It was observed that, very few reports are available on the study of nonlinear functional differential equations in Banach space. We referred the hybrid fixed point theorems in this study [4-6, 9]. The FONFDE (1.1) contributes to the area of functional differential equations, newly.

2. Preliminaries and Definitions

Definition (2.1): The Banach algebra is X with norm $\|.\|$. An operator $A : X \to X$ is called $\mathcal{D}-$Lipschitzian if $\exists\ \mu : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\|Ax - Ay\| \le \mu\|x - y\| \qquad (2.2)$$

where, μ is the continuous non-decreasing function for all $x, y \in X$ with $\mu(0) = 0$. When $\mu(r) = \alpha r$, $r > 0$, A is called as Lipschitzian with Lipschitz constant α. If $\alpha < 1$, A is contraction with contraction constant α. If $\mu(r) < r$ for all $r > 0$ then A is a nonlinear contraction on X.

Definition (2.3): An operator $T : X \to X$ is called compact if $\overline{T(S)}$ is compact for any subset S of X. Similarly $T : X \to X$ is called totally bounded if T maps a bounded subset of X into the totally bounded subset of X, finally $T : X \to X$ is called completely continuous operator if it is continuous and totally bounded operator on X. Note that every compact operator is totally bounded but converse may not be true. We use the following hybrid theorem as tool for proving the existence of solution of FONFDE (1.1).

Theorem (2.4): (Dhage [7]) Let X be a Banach algebra and let $A, B : X \to X$ be operators satisfying
(a) A is a $\mathcal{D} -$ Lipschitzian with a $\mathcal{D} -$ function μ,
(b) B is compact and continuous, and
(c) $M\mu(r) < r$ whenever $r > 0$, where $M = \|B(X)\| = sup\{\|Bx\|: x \in X\}$.
Then either
(i) the equation $\lambda A\left(\frac{x}{\lambda}\right) Bx = x$ has a solution for $\lambda = 1$, or
(ii) the set $\xi = \left\{u \in X: \lambda A\left(\frac{u}{\lambda}\right) Bu = u,\ 0 < \lambda < 1\right\}$ is unbounded.

Corollary: Let X be Banach algebra and let $A, B : X \to X$ be operators satisfying
(d) A is a $\mathcal{D} -$ Lipschitzian with a Lipschitz constant α
(e) B is compact and continuous, and
(f) $M\alpha < r$ whenever $r > 0$, $M = \|B(X)\| = sup\{\|Bx\|: x \in X\}$.
Then either
(j) The equation $\lambda A\left(\frac{x}{\lambda}\right) Bx = x$ has a solution for $\lambda = 1$, or
(ii) The set $\xi = \left\{u \in X: \lambda A\left(\frac{u}{\lambda}\right) Bu = u,\ 0 < \lambda < 1\right\}$ is unbounded.

3. Main Result

Let $M(J, \mathbb{R})$ and $B(J, \mathbb{R})$ indicate the spaces of measurable and bounded real-valued functions on J, respectively. We can deduce the existence of FONFDE (1.1) solution in the space $C(J, \mathbb{R})$. Norm $\|.\|$ and multiplication '.' in space $C(J, \mathbb{R})$ defined by

$\|x\| = \sup_{t\in J}|x(t)|$, $(x.y)(t) = x(t).y(t)$, $t \in J$ respectively.

Then by these norm and multiplication, $C(J, \mathbb{R})$ became Banach algebra. The following definitions are useful in the study.

Definition (3.1): A function $\gamma: I_+ \times C \times \mathbb{R} \to \mathbb{R}$ is said to be satisfied a condition of $L^1_X -$ Caratheodory if
1. $t \to \gamma(t, x_t, x)$ is measurable for each $x_t \in C$.
2. $x \to \gamma(t, x_t, x)$ is continuous at everywhere $t \in I_+$,
3. There exists $h \in L^1(I_+, \mathbb{R})$ viz. $|\gamma(t, x_t, x) \le h(t)|$ at everywhere $t \in I_+$ for all $x_t \in . C$.

Following hypotheses are needed in the study.

(A_1) A continuous function $f: I_+ \times\to \mathbb{R} - \{0\}$ and $\exists$ a function $k \in B(I_+, \mathbb{R})$ viz. $k(t) > 0$ at everywhere $t \in I_+$ and

$|f(t, x) - f(t, y)| \le k(t)|x - y|$ for all $x, y \in \mathbb{R}$.
(A_2) $f\big(0, \phi(0)\big) = 1$

(A_3) The function g is L_X^1 − Caratheodory with bound function h.
(A_4) $\exists$ a function $\Phi: [0,\infty) \to (0,\infty)$ which is continuous and non-decreasing and $\delta \in L^1(I_+,\mathbb{R})$ viz. $\delta(t) > 0$ at everywhere $t \in J$ and

$$|g(t,x_t,x)| \le \delta(t)\Phi(\|x\|_C),\ \text{ a.e. } t \in I_+,\ \text{ for all } x_t \in C.$$

Theorem (3.2): The hypotheses are assumed to be (A_1) - (A_4) hold. Suppose that

$$\int_{C_1}^{\infty} \frac{ds}{\Phi(s)} > C_2\|\delta\|_{L^1} \qquad (3.3)$$

where $C_1 = \frac{F\|\phi\|_C}{1-\|k\|(\|\phi\|_C+\|h\|_{L^1})}$, $C_2 = \frac{F}{1-\|k\|(\|\phi\|_C+\|h\|_{L^1})}$, $\|k\|(\|\phi\|_C + \|h\|_{L^1}) < 1$,

$F = \max_{t\in J}|f(t,0)|$, and $\|k\| = \max_{t\in J}|k(t)|$. Then the FONFDE (1.1) has a solution on J.

Proof: Now, convert the FONFDE (1.1) into an equivalent nonlinear functional integral equation (in short NFIE (3.4)) as

$$x(t) = [f(t,x(t))]\left(\phi(0) + \int_0^t g(s,x_s,xs)ds\right).\quad \text{if} \quad t \in I_+ \qquad (3.4)$$

and

$$x(t) = \phi(t)\,,\ \text{ if }\ t \in I \qquad (3.5)$$

The operators A and B are defined on $C(J,\mathbb{R})$ by

$$Ax(t) = \begin{cases} f(t,x(t)) & if\ t \in I_+ \\ 1 & if\ t \in I \end{cases} \qquad (3.6)$$

and

$$Bx(t) = \begin{cases} \phi(0) + \int_0^t g(t,x_t,x(t))ds & if\ if\ t \in I_+ \\ \phi(t) & if\ t \in I \end{cases} \qquad (3.7)$$

Define the operators $A, B : C(J,\mathbb{R}) \to C(J,\mathbb{R})$. The FONFDE (1.1) is equivalent to the operator equation

$$x(t) = Ax(t)Bx(t), \quad t \in J \qquad (3.8)$$

We have shown that, the operators A and B satisfied all the hypotheses of corollary of theorem (2.4), as below:
Now, it is needed to show, A is a Lipschitz on $C(J,\mathbb{R})$. Let $x, y \in C(J,\mathbb{R})$, then (H_1),

$$\begin{aligned} \Big|Ax(t) - Ay(t) &\le \big|f(t,x(t)) - f(t,y(t))\big|\Big| \\ &\le k(t)\,|x(t) - y(t)| \\ &\le k\|x - y\| \end{aligned}$$

for all $t \in J$. Taking the supremum over t we obtain

$$\|Ax - Ay\| \le \|k\|\|x - y\|$$

.
for all $x, y \in C(J,\mathbb{R})$. Hence, A is a Lipschitz on $C(J,\mathbb{R})$ with a constant $\|k\|$. Now, again it is needed to show that B is completely continuous on $C(J,\mathbb{R})$. In Granas et al. [11] it is shown that . the operator B is

continuous on $C(J,\mathbb{R})$. Let $S \subseteq C(J,\mathbb{R})$. It has shown below that $B\left(C(J,\mathbb{R})\right)$ is equicontinuous and uniformly bounded set in $C(J,\mathbb{R})$. Since $g\left(t,x_t,x(t)\right)$ is L_X^1-Caratheodory, we have,

$$|Bx(t)| \le \|\phi\|_C + \int_0^t \left|g\left(s,x_s,x(s)\right)\right| ds$$
$$\le \|\phi\|_C + \int_0^t h(s)ds$$
$$\le \|\phi\|_C + \|h\|_{L^1}$$

We obtained $|Bx| \le M$ for all $x \in S$, where $M = \|\phi\|_C + \|h\|_{L^1}$ after taking the supremum over t. Therefore, $B\left(C(J,\mathbb{R})\right)$ is uniformly bounded set in $C(J,\mathbb{R})$. We need to show that $B\left(C(J,\mathbb{R})\right)$ is equicontinuous set in $C(J,\mathbb{R})$. Let $t,\sigma \in I_+$ then for $x \in C(J,\mathbb{R})$ by (3.7),

$$|Bx(t) - Bx(\sigma)| \le \left|\int_0^t g\left(s,x_s,x(s)\right)ds - \int_0^t g\left(s,x_s,x(s)\right)ds\right|$$
$$\le \left|\int_\sigma^t g\left(s,x_s,x(s)\right)ds\right|$$
$$\le \int_\sigma^t h(s)ds$$
$$\le |p(t) - p(\sigma)|$$

where $p(t) = \int_0^t h(s)ds$.

$$\therefore \quad |Bx(t) - Bx(\sigma)| \to 0 \quad for \quad t \to \sigma.$$

Let $\sigma \in I$, $t \in I_+$, then we get,

$$|Bx(t) - Bx(\sigma)| \le |\phi(\sigma) - \phi(0)| + \left|\int_\sigma^t g\left(s,x_s,x(s)\right)ds\right|$$
$$\le |\phi(\sigma) - \phi(0)| + \int_\sigma^t h(s)ds$$
$$\le |\phi(\sigma) - \phi(0)| + |p(t) - p(\sigma)|$$

Also if $t,\sigma \in I_+$, we get

$$|Bx(t) - Bx(\sigma)| \le |\phi(t) - \phi(\sigma)|$$

$\therefore$ $|Bx(t) - Bx(\sigma)| \to 0$ as $\sigma \to t$ for all $\sigma, t \in J$.

Therefore, $B\left(C(J,\mathbb{R})\right)$ is an equicontinuous set and then it is relatively compact by Arzela-Ascoli theorem. Therefore B is a continuous and compact operator on $C(J,\mathbb{R})$. In this way, all the conditions of theorem (2.4) are satisfied and either conclusion (i) or (ii) holds good. It is observed that the conclusion (i) is applicable and not (ii). Let $x \in X$ be any solution to The FONFDE (1.1). Then for any $\lambda \in (0,1)$,

$$x(t) = \lambda A\left(\frac{x}{\lambda}\right)(t)Bx(t)$$

$$= \begin{cases} \lambda\left[f\left(t,\frac{x(t)}{\lambda}\right)\right]\left(\phi(0) + \int_0^t g(s,x_s,x(s))ds\right), & t \in I_+ \\ \lambda\phi(t), & t \in I \end{cases}$$

For $t \in J$. Then we have

$$x(t) \leq \lambda\left|f\left(s,\frac{x(t)}{\lambda}\right)\right|\left(\|\phi\|_C + +\left|\int_0^t g(s,x_s,x(s))ds\right|\right)$$

$$\leq \lambda\left(\left|f\left(s,\frac{x(t)}{\lambda}\right) - f(t,0)\right| + |f(t,0)|\right)\left(\|\phi\|_C + +\left|\int_0^t g(s,x_s,x(s))ds\right|\right)$$

$$\leq [k(t)|x(t)| + F]\left(\|\phi\|_C + +\left|\int_0^t g(s,x_s,x(s))ds\right|\right)$$

$$\leq k(t)\|x\|\left(\|\phi\|_C + +\left|\int_0^t g(s,x_s,x(s))ds\right|\right) + F\left(\|\phi\|_C + +\left|\int_0^t g(s,x_s,x(s))ds\right|\right)$$

$$\leq \|k\|\|x\|(\|\phi\|_C + \|h\|_{L^1}) + F\|\phi\|_C + F\int_0^t \delta(s)\Phi(\|x_s\|_C)ds \qquad (3.9)$$

Put $\ell(t) = \sup_{s\in[-r,t]}|x(s)|$, for $t \in J$. Then

$|x(t)| \leq \ell(t)$, for every $t \in J$ and $\|x_t\|_C \leq \ell(t)$ for every $t \in I_+$,

Hence, a point $t^* \in [-r,t]$ such that $\ell(t) = |x(t^*)| \implies$

$$\ell(t) = |x(t^*)|$$

$$\leq \|k\||x(t^*)|(\|\phi\|_C + \|h\|_{L^1}) + F\left(\|\phi\|_C + \int_0^{t^*} \delta(s)\Phi(\|x_s\|_C)ds\right)$$

$$\leq \|k\|\ell(t)(\|\phi\|_C + \|h\|_{L^1}) + F\left(\|\phi\|_C + \int_0^{t^*} \delta(s)\Phi(\ell(s))ds\right)$$

$$\ell(t) - \|k\|\ell(t)(\|\phi\|_C + \|h\|_{L^1}) \leq F\|\phi\|_C + F\int_0^{t^*} \delta(s)\Phi(\ell(s))ds$$

$$\ell(t)\left(1 - \|k\|(\|\phi\|_C + \|h\|_{L^1})\right) \leq F\|\phi\|_C + F\int_0^{t^*} \delta(s)\Phi(\ell(s))ds$$

$$= C_1 + C_2\int_0^{t^*} \delta(s)\Phi(\ell(s))ds \qquad (3.10)$$

Where $C_1 = \frac{F\|\phi\|_C}{1-\|k\|(\|\phi\|_C+\|h\|_{L^1})}$ and $C_2 = \frac{F}{1-\|k\|(\|\phi\|_C+\|h\|_{L^1})}$

Let
$q(t) = C_1 + C_2 \int_0^{t^*} \delta(s)\Phi(\ell(s))ds,$

Then $\ell(t) \leq q(t)$ and a direct differentiation of $q(t)$ yields

$$q'(t) \leq C_2\delta(t)\Phi(q(t)) \quad \text{and} \quad q(0) = C_1 \tag{3.11}$$

i.e

$$\int_o^t \frac{q'(t)}{\Phi(q(s))} \leq C_2 \int_0^{t^*} \delta(s)ds \leq C_2\|\delta\|_{L^1}$$

In the above integral, by changing the variables, we get,

$$\int_{C_1}^{q(t)} \frac{ds}{\Phi(s)} \leq C_2\|\delta\|_{L^1} < \int_{C_1}^{\infty} \frac{ds}{\Phi(s)}$$

Mean value theorem yields $M > 0$ such that $q(t) \leq M$ for all $t \in J$
$\Rightarrow |x(t)| \leq \ell(t) \leq q(t) \leq M$ for all $t \in J$.
Therefore, conclusion (ii) does not hold. Therefore the operator equation $AxBx = x$ and consequently the FONFDE (1.1) has a solution on J. Hence proved.

4. An Example

Given the closed and bounded interval $I = [-\frac{\pi}{2}, 0]$ and $I_+ = [0, \frac{\pi}{2}]$ in $\mathbb{R}$. Consider the FONFDE

$$\left(\frac{x(t)}{f(t,x(t))}\right)' = \frac{p(t)}{(1+\|x_t\|_C)e^t} \quad \text{a.e.} \quad t \in I_+$$

$x(t) = sint \quad t \in I,$ where $p \in L^1(I_+, \mathbb{R})$ and $f: I_+ \times\to \mathbb{R} - \{0\}$ is defined by

$$f(t, x(t)) = 1 + \alpha|x(t)| \quad , \qquad \alpha > 0$$

$\forall\, t \in I_+$. obviously $f: I_+ \times\to \mathbb{R} - \{0\}$, A function $g: I_+ \times C \times \mathbb{R} \to \mathbb{R}$ is defined by $g(t, x_t, x(t)) = \frac{p(t)}{(1+\|x_t\|_C)e^t}$. It is verified simply that, f is continuous and Lipschitz on $J \times \mathbb{R}$ with a Lipschitz constant α. Further $g(t, x_t, x(t))$ is $L_X^1 -$ Caratheodory with bound function $h(t) = p(t)$ on I_+. Therefore $\alpha(1 + \|p\|_{L^1}) < 1$, then by main theorem (3.3) the FONFDE (4.1) has a solution on $J = I \cup I_+$.

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