International Journal for Multidisciplinary Research (IJFMR)
E-ISSN: 2582-2160 ● Website: www.ijfmr.com ● Email: editor@ijfmr.com

Techniques on finding Integer Solutions to Hyperboloid of Two Sheets $x^2 - 6xy + y^2 + 6x - 2y + 1 = z^2$

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Abstract
In this paper, the Diophantine equation representing hyperboloid of two sheets given by $x^2 - 6xy + y^2 + 6x - 2y + 1 = z^2$ is considered and analyzed for its non-zero distinct solutions in integers. Varieties of solution patterns are determined through substitution technique and factorization method.

Keywords: Hyperboloid of two sheets, Ternary quadratic equation, Non- Homogeneous quadratic equation, Integer solutions

Introduction
It is quite obvious that Diophantine equations, one of the areas of number theory, are rich in variety [1, 2]. In particular, the ternary quadratic Diophantine equations in connection with geometrical figures occupy a pivotal role in the orbit of mathematics and have a wealth of historical significance. In this context, one may refer [3-11] for second degree Diophantine equations with three unknowns representing different geometrical figures.
In this paper, the ternary quadratic Diophantine equation representing hyperboloid of two sheets given by $x^2 - 6xy + y^2 + 6x - 2y + 1 = z^2$ is studied for determining its integer solutions successfully through substitution strategy and method of factorization.

Method of analysis
The ternary quadratic Diophantine equation representing hyperboloid of two sheets to be solved is given by

$$x^2 - 6xy + y^2 + 6x - 2y + 1 = z^2 \tag{1}$$

The process of obtaining varieties of solution patterns is illustrated below:

Technical procedure-1
Treating (1) as a quadratic in y and solving for the same, one has

$$y = 3x + 1 \pm \sqrt{z^2 + 8x^2} \qquad (2)$$

The square-root on the R.H.S. of (2) is removed when

$$x = 2rs \,, z = 8r^2 - s^2 \qquad (3)$$

Substituting (3) in (2), we have

$$y = 6rs + 1 \pm (8r^2 + s^2) \qquad (4)$$

Thus, (3) & (4) represent two sets of integer solutions to (1).

Note 1

To remove the square-root on the R.H.S. of (2), the following process may be utilized :

Let

$$\alpha^2 = z^2 + 8x^2$$

which is expressed as the system of double equations as in Table 1 below :

Table 1-System of double equations

System	I	II	III	IV
$\alpha + z$	$4x^2$	$2x^2$	x^2	$8x$
$\alpha - z$	2	4	8	x

Solving each of the above four systems, the values of x, z are obtained .From (2), the corresponding y values are obtained. For simplicity, the integer solutions satisfying (1) are presented below:

Solutions from System 1

$$x = s \,, y = 3s + 1 \pm (2s^2 + 1), z = (2s^2 - 1)$$

Solutions from System 1I

$$x = s \,, y = 3s + 1 \pm (s^2 + 2), z = (s^2 - 2)$$

Solutions from System III

$$x = 2s \,, y = 6s + 1 \pm (2s^2 + 4), z = (2s^2 - 4)$$

Solutions from System IV

$$x = 2s \,, y = 6s + 1 \pm (9s), z = 7s$$

Technical procedure-2

Treating (1) as a quadratic in x and solving for the same, one has

$$x = 3\,(y-1) \pm \sqrt{z^2 + 8(y-1)^2} \qquad (5)$$

The square-root on the R.H.S. of (5) is removed when

$$y = 2rs + 1 \,, z = 8r^2 - s^2 \qquad (6)$$

Substituting (6) in (5), we have

$$x = 6rs \pm (8r^2 + s^2) \qquad (7)$$

Thus, (6) & (7) represent two sets of integer solutions to (1).

Note 2

Consider

$$Y = y - 1 \qquad (8)$$

Let

$$\alpha^2 = z^2 + 8Y^2$$

which is expressed as the system of double equations as in Table 2 below :

Table 2-System of double equations

System	I	II	III	IV	V
$\alpha + z$	$4Y^2$	$2Y^2$	Y^2	$8Y$	$4Y$
$\alpha - z$	2	4	8	Y	$2Y$

Solving each of the above five systems, the values of Y, z are obtained .From (8) & (5), the corresponding y, x values are obtained. For simplicity, the integer solutions satisfying (1) are presented below:

Solutions from System I

$$z = 2s^2 - 1 , y = s + 1, x = 3s \pm (2s^2 + 1)$$

Solutions from System II

$$z = s^2 - 2 , y = s + 1, x = 3s \pm (s^2 + 2)$$

Solutions from System III

$$z = 2s^2 - 4 , y = 2s + 1, x = 6s \pm (2s^2 + 4)$$

Solutions from System IV

$$z = 7s , y = 2s + 1, x = 6s \pm (9s)$$

Solutions from System V

$$z = s , y = s + 1, x = 6s$$

Technical procedure-3

The substitution of the transformations

$$x = u + v , y = u - v , u \neq v \neq 0 \tag{9}$$

in (1) leads to the ternary quadratic equation

$$z^2 + P^2 = 2Q^2 \tag{10}$$

where

$$P = 2u - 1 , Q = 2v + 1 \tag{11}$$

Express (10) in the form of ratio as

$$\frac{Q + z}{P + Q} = \frac{P - Q}{Q - z} = \frac{\alpha}{\beta} , \beta \neq 0$$

Solving the above system of double equations through the method of cross-multiplication, we get

$$P = \beta^2 + 2\alpha\beta - \alpha^2 , Q = \beta^2 + \alpha^2 , \tag{12}$$

and

$$z = \alpha^2 + 2\alpha\beta - \beta^2 . \tag{13}$$

Substituting (12) in (11) and applying (2), we have

$$x = \beta^2 + \alpha\beta , y = \alpha\beta - \alpha^2 + 1 . \tag{14}$$

Thus, (13) & (14) represent the integer solutions to (1).

Note 3

One may also consider (10) as

$$\frac{Q+z}{P-Q}=\frac{P+Q}{Q-z}=\frac{\alpha}{\beta},\beta\neq 0$$

In this case, the corresponding integer solutions to (1) are found to be

$$x=\alpha^2+\alpha\beta\ ,y=\alpha\beta-\beta^2+1.z=\alpha^2-2\alpha\beta-\beta^2$$

Technical procedure-4

Now, we apply the factorization method to solve (10).

Let

$$Q=25\,(a^2+b^2) \tag{15}$$

Consider the integer 2 on the R.H.S. of (10) as the product of complex conjugates as shown below:

$$2=\frac{(7+i)\,(7-i)}{25} \tag{16}$$

Substituting (15) & (16) in (10) and applying the factorization method, consider

$$z+iP=5\,(7+i)\,(a+ib)^2 \tag{17}$$

On equating the real and imaginary parts in (17) , we get

$$z=5[7(a^2-b^2)-2ab] \tag{18}$$

and

$$P=5[(a^2-b^2)+14ab]$$

In view of (11) & (9), we have

$$x=15a^2+10b^2+35ab\ ,y=-10a^2-15b^2+35ab+1\,. \tag{19}$$

Thus, (18) & (19) represent the integer solutions to (1).

Note 4

Apart from (16), one may also consider

$$2=\frac{(1+7i)\,(1-7i)}{25}\ ,$$

$$2=(1+i)\,(1-i)$$

The repetition of the above process leads to different sets of integer solutions to (1).

Technical procedure-5

Write (10) as

$$2Q^2-P^2=z^2*1 \tag{20}$$

Let

$$z=2\,a^2-b^2 \tag{21}$$

Consider the integer 1 on the R.H.S. of (20) as

$$1=(\sqrt{2}+1)\,(\sqrt{2}-1) \tag{22}$$

Substituting (21) & (22) in (20) and applying the factorization method, consider

$$\sqrt{2}Q+P=(\sqrt{2}+1)\,(\sqrt{2}\,a+b)^2 \tag{23}$$

On equating the rational and irrational parts in (23), we get

$$P = [2a^2 + b^2 + 4ab], \quad Q = [2a^2 + b^2 + 2ab] \tag{24}$$

In view of (11) ,we have

$$u = a^2 + 2ab + \frac{(b^2+1)}{2}, v = a^2 + ab + \frac{(b^2-1)}{2}$$

In view of (9), we get

$$x = 2a^2 + 3ab + b^2, y = ab + 1. \tag{25}$$

Thus, (25) & (21) represent the integer solutions to (1).

Note 5

Apart from (22), one may also consider

$$1 = (5\sqrt{2} + 7)(5\sqrt{2} - 7)$$

The repetition of the above process leads to a different set of integer solutions to (1).

Technical procedure-6

Taking

$$Q = X - T, P = X - 2T \tag{26}$$

in (10) ,it gives

$$X^2 = 2T^2 + z^2 \tag{27}$$

which is satisfied by

$$T = 2rs, X = 2r^2 + s^2 \tag{28}$$

and

$$z = 2r^2 - s^2 \tag{29}$$

Using (28) in (26), we have

$$Q = 2r^2 + s^2 - 2rs, P = 2r^2 + s^2 - 4rs$$

In view of (11) & (9), we get

$$x = 2r^2 - 3rs + s^2, y = 1 - rs \tag{30}$$

Thus, (30) & (29) represent the integer solutions to (1).

Technical procedure-7

Choosing

$$Q = 2X + 1, z = X + 4 \tag{31}$$

in (10) ,it gives

$$P^2 = 7X^2 - 14 \tag{32}$$

whose least positive integer solution is

$$X_0 = 3, P_0 = 7$$

To obtain the other solutions to (32), consider the pellian equation

$$P^2 = 7X^2 + 1$$

whose general solution $(\bar{X}_n, \bar{P}_n)$ is given by

$$\bar{X}_n = \frac{g_n}{2\sqrt{7}}, \bar{P}_n = \frac{f_n}{2}$$

where

$f_n = (8+3\sqrt{7})^{n+1} + (8-3\sqrt{7})^{n+1}, g_n = (8+3\sqrt{7})^{n+1} - (8-3\sqrt{7})^{n+1}$

Applying the lemma of Brahmagupta between $(X_0, P_0) \& (\bar{X}_n, \bar{P}_n)$, the general solutions to (32) is given by

$$X_{n+1} = \frac{(3f_n + \sqrt{7}g_n)}{2}, P_{n+1} = \frac{(7f_n + 3\sqrt{7}g_n)}{2}$$

From (31),we have

$$Q_{n+1} = 2X_{n+1} + 1 = (3f_n + \sqrt{7}g_n) + 1$$

and

$$z_{n+1} = X_{n+1} + 4 = \frac{(3f_n + \sqrt{7}g_n)}{2} + 4 \qquad (33)$$

In view of (11), we have

$$u_{n+1} = \frac{(7f_n + 3\sqrt{7}g_n + 2)}{4}, v_{n+1} = \frac{(3f_n + \sqrt{7}g_n)}{2}$$

From (9), we have

$$x_{n+1} = \frac{(13f_n + 5\sqrt{7}g_n + 2)}{4}, y_{n+1} = \frac{(f_n + \sqrt{7}g_n + 2)}{4} \qquad (34)$$

Thus, (33) & (34) represent the integer solutions to (1).

Conclusion:

In this paper ,we have presented varieties of integer solutions on the hyperboloid of two sheets represented by the non-homogeneous ternary quadratic Diophantine equation given in title. As the choices of ternary quadratic equations are rich in variety , the readers of this paper may search for other representations to hyperboloid of two sheets or one sheet in obtaining patterns of integer solutions successfully.

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