International Journal for Multidisciplinary Research (IJFMR)
E-ISSN: 2582-2160 ● Website: www.ijfmr.com ● Email: editor@ijfmr.com

A Review of Brahmagupta's Contributions

S.R. Singh[1], Ajay Kumar[2], Anuradha Singh[3], Rashmi Yadav[4]

[1]Department of Mathematics, Chaudhary Charan Singh University, Meerut
[2]Bharat Institute of Technology, Meerut,
[3]RKGIT, Meerut
[4]Department of Mathematics, Vardhman College Bijnor

Abstract:
The life, accomplishments, and legacy of prominent Indian mathematician and astronomer Brahmagupta from the 7th century are examined in this review. Brahmagupta, who is well-known for his contributions to algebra, arithmetic, and astronomy, provided ground-breaking ideas in his "Brāhmasphuṭasiddhānta,"a foundational work that included the rules for mathematical operations involving zero and negative integers. His efforts established the groundwork for contemporary mathematics, inspiring other researchers in mediaeval Europe as well as the Islamic world. In order to emphasise Brahmagupta's methods, his contribution to the advancement of mathematical thought, and the ongoing influence of his work on current mathematical practices, this review synthesises historical accounts and contemporary interpretations.

*"As the sun eclipses the stars by his brilliance, so does the man of knowledge eclipses the fame of others in assemblies of people if he proposes Algebraic problems, and still more if he solves them. A person who can, within a year, solve $x^2 - 92y^2 = 1$ is a mathematician." -**Brahmagupta***

Introduction*:*
Brahmagupta holds a unique position in the history of Ancient Indian Mathematics. He contributed such elegant results to Geometry and Number Theory that today's mathematicians still marvel at their originality. His theorems leading to the calculation of the circumradius of a triangle and the lengths of the diagonals of a cyclic quadrilateral, construction of a rational cyclic quadrilateral, and integer solutions to a single second-degree equation are certainly the hallmarks of a genius. Brahmagupta, a most accomplished mathematician, lived during this medieval period and was responsible for creating good mathematics in the form of geometrical theorems and number-theoretic results. This is besides his contribution to astronomy. He was born in a village called Bhillamala in NorthWestRajastan, then the capital of the Gurjara-Pratihara dynasty in the year 598 AD. during the reign of the Chavda dynasty ruler, Vyagrahamukha. He was the son of Jishnugupta and was a Hindu by religion, in particular, a Shaivite. He was born into a mathematical family, as his father was an astrologer. Brahmagupta considered himself to be an astrologer like his father rather than a mathematician. He relied more on his religion to be his guide in his discoveries. Many of his proofs and claims of the world in mathematics are written as poetry, yet they are still accounted as proofs and claims. He was certainly a mathematician of preeminence for his times, but he also had the habit of criticizing his predecessors sharply for some of their faults and omissions. He was the head of the Ujjain observatory. He passed away in the year 668 AD.

Works of Brahmagupta:

His works included astronomy, gravity theory, negative numbers, the use of zero, quadratic equations, and square roots. Most of the information published by Brahmagupta is found in ancient texts, such as the Brāhmasphuṭasiddhānta, twenty-five chapters long. The meaning behind Brāhmasphuṭasiddhānta is "the system of the god of creation and astronomy". Within the first ten chapters of the book, Brahmagupta wrote things common to the mathematical and astronomical developments at the time. They had topics on mean longitudes of planets, true longitudes of planets, diurnal rotation, lunar eclipses, solar eclipses, risings and settings of the sun/moon, the phases of the moon, and the conjunctions of planets with the stars and other planets. The second half of the book holds major insight into the world. It holds explanations of the first half, algebra, observations, and calculations on the calendar, meters, spheres, and instruments. One well-known explanation of algebra Brahmagupta explained in Brāhmasphuṭasiddhānta was multiplication using a place-vale system. His other works were the Cadamekela, the Khandakhadyaka, and the Durkeamynarda which consists of works written in only verses and no proofs. His most famous and well-known contribution to the mathematics world is his definition and use of zero with negative numbers. The truths of the universe could not have been discovered, written, and understood by mathematicians without Brahmagupta's concept of zero. He did create a set of rules for zero as follows:

A debt minus zero is a debt.
A fortune minus zero is a fortune.
Zero minus zero is a zero.
A debt subtracted from zero is a fortune.
A fortune subtracted from zero is a debt.
The product of zero multiplied by a debt or fortune is zero.
The product of zero multiplied by zero is zero.
The product or quotient of two fortunes is one fortune.
The product or quotient of two debts is one fortune.
The product or quotient of a debt and a fortune is a debt.
The product or quotient of a fortune and a debt is a debt.

In other words, Brahmagupta's rules for zero is as follows:

- A positive number minus zero is the same positive number.
- A negative number minus zero is the same negative number.
- Zero minus zero is zero.
- A negative number subtracted from zero is its opposite value (a positive number).
- A positive number subtracted from zero is its opposite value (a negative number).
- A number, positive or negative, multiplied by zero is zero. Zero multiplied by zero is zero.
- The product or quotient of two positive numbers is a positive number.
- The product of the quotient of two negative numbers is a positive number.
- The product or quotient of a positive and negative number, or vice versa, is a negative number.

Some of his rules of zero did end up being incorrect, such as the fact that zero divided by zero is zero. Brahmagupta also worked on the rules and solutions for arithmetic sequences, quadratic equations with real roots, and infinity, and contributed to the works of Pell's Equation. Brahmagupta was fascinated by arithmetic equations and gave the formulas for finding the sum of squares and cubes to the nth integer. He also computed an algorithm for square roots equivalent the Newton-Raphson Iterative Formula.

Brahmagupta's Algorithm for Square Roots

$$x^{\frac{1}{2}} = (x-y)(x+y) + y^{\frac{1}{2}}$$

The Newton-Raphson Iteration

$$x_{n+1} = x_1 - \frac{f(x_n)}{f'(x_n)}$$

Another famous work of his was finding the formulas for the area of a triangle and cyclic quadrilateral in terms of sides. Area of a Triangle = √ s (s − a) (s − b) (s − c) where s is the semi perimeter and a, b, c are the sides of the triangle.

Area of a Cyclic Quadrilateral √ s (s − a) (s − b) (s − c) (s − d), where s is the semi perimeter and a, b, c, d are the sides of the quadrilateral. The semi-perimeter is equal to the sum of the number of sides of the figure divided by two. From finding the area of a cyclic quadrilateral, Brahmagupta's Theorem was established which states that: In a cyclic quadrilateral whose diagonals are perpendicular to each other, the line through the point of intersection of the diagonals which is perpendicular to one side bisects the opposite side.

Contribution of Brahmagupta in Mathematics:

Brahmagupta made fundamental contributions to many branches of mathematics.

- Brahmagupta–Fibonacci identity (Algebra) expresses the product of two sums of two squares as a sum of two squares in two different ways. Hence the set of all sums of two squares is closed under multiplication. Specifically, the identity says

$$\begin{aligned} \left(a^2+b^2\right)\left(c^2+d^2\right) &= (ac-bd)^2+(ad+bc)^2 \qquad (1) \\ &= (ac+bd)^2+(ad-bc)^2. \qquad (2) \end{aligned}$$

- Cyclic Quadrilaterals (Geometry)

A cyclic quadrilateral is a quadrilateral whose 4 vertices lie on a circle. If a, b, c, d are the sides of such a quadrilateral, Brahmagupta's formula for its area is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$,where s is the semi perimeter, namely, s = (a + b + c + d)/2. If we take a = 0 this specializes into the ancient formula for the area of a triangle, due to Heron.

He also derived the formulae for the lengths m, n of the two diagonals for such a quadrilateral, namely,

m^2 = (ab + cd) (ac + bd) / (ad + bc),

n^2 = (ac + bd) (ad + bc) / (ab + cd).

Integer cyclic quadrilaterals: Brahmagupta gave a simple method to construct cyclic quadrilaterals with integer sides, integer diagonals, and integer areas. Then we have a cyclic quadrilateral ABCD with integer sides bz; cy; az; cx and integer diagonals ay + bx; ax+by and integer area 1/2 {(ax+by)(ay+bx)}.

- Brahmagupta's work on Pell's equation: Structure of the set of solutions: Discovered that there is an infinity of solutions and that the set of solutions forms a commutative group in the modern sense. The key to this is to define a suitable multiplication for pairs of solutions which will give a third solution. (x, y) = (a, b) ∗ (c, d); x = ac+Nbd, y = ad+bc. This follows from Brahmagupta's identity: $(a^2 - N b^2)(c^2 - N d^2) = x^2 - Ny^2$
- *Construction of an infinity of solutions if one is known* Thus, if we have one solution (p, q), we can construct an infinity of solutions (p_k, q_k) (k = 0, 1, 2, . . ., (p_0, q_0) = (p, q) by the following recursion formula $p_k = pp_{k-1} + Nqq_{k-1}$, $qk = qp_{k-1} + pq_{k-1}$.

He also knew (in many cases at least) how to get a minimal solution in a finite number of steps, starting from some (p, q) such that $p^2 - Nq^2 = m$ where m is a small integer.

A Brief Explanation of Brahmagupta's Books or Works:

Brahmagupta wrote his first book, Brāhmasphuṭasiddhānta, (the Opening of the Universe) in the year 628 AD. This book contains 1008 slokas(verses) in 24 chapters and deals with astronomy, arithmetic, algebra, geometry, and number theory. He was the first to introduce zero as a digit. This was translated into Arabic with the title Sind Hind. Brāhmasphuṭasiddhānta, is composed in elliptic verse, as was common practice in Indian mathematics, and subsequently has a poetic ring to it. As no proofs are given, it is not known how Brahmagupta's mathematics was derived, In the Brāhmasphuṭasiddhānta, among the major developments are those in the areas of:

Arithmetic:

Brahmagupta possessed a greater understanding of the number system (and the place value system) than anyone to that point. At the beginning of chapter 12, entitled Calculation, he details operations on fractions. He explains how to find the cube and cube root of an integer and later gives rules facilitating the computation of squares and square roots. He then gives rules for dealing 5 types of combinations of fractions:

$$\frac{a}{c}+\frac{b}{c}, \frac{a}{c}.\frac{b}{d}, \frac{a}{1}+\frac{b}{d}, \frac{a}{c}+\frac{b}{d}.\frac{a}{c}=\frac{a(b+d)}{cd}, \frac{a}{c}-\frac{b}{d}.\frac{a}{c}=\frac{a(d-b)}{cd}$$

Brahmagupta then goes on to give the sum of the squares and cubes of the first n integers. He found the result in terms of the sum of the first n integers rather than in terms of n as is the modern practice. He gives the sum of the squares of the first *n* natural numbers as $\frac{n(n+1)(2n+1)}{6}$ and the sum of the cubes of the first n natural numbers as $\frac{(n(n+1))^2}{2}$.

Brahmagupta made use of an important concept in mathematics, the number 'zero'. The Brāhmasphuṭasiddhānta, is the earliest known text to treat zero as a number in its own right, rather than as simply a placeholder digit in representing another number as was done by the Babylonians or as a symbol for lack of quantity as was done by Ptolemy and the Romans. In chapter 19, he describes operations on negative numbers. He also describes addition, subtraction, and multiplication. Brahmagupta was the first to attempt to divide by zero, and while his attempts of showing $n/0 = \infty$ were not ultimately successful they demonstrate an advanced understanding of an extremely abstract concept.

Algebra:

Brahmagupta gave the solution of the general linear equation ax + c = in chapter 18. He further gave two equivalent solutions to the general quadratic equation which are respectively: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

He went on to solve systems of simultaneous indeterminate equations stating that the desired variable must first be isolated, and then the equation must be divided by the desired variable's coefficient. The algebra of Brahmagupta is syncopated.

Brahmagupta went on to give a recurrence relation for generating solutions to certain instances of Diophantine equations of the second degree such as $Nx^2 + 1 = y^2$ by using the Euclidean algorithm. The

Euclidean algorithm was known to him as "Kuttaka" since it breaks numbers down into even smaller pieces. The key to his solution was the identity:

$$(x_1^2-Ny_1^2).\,(x_2^2-Ny_2^2)=(x_1x_2+Ny_1y_2)^2-N(x_1y_2+x_2y_1)^2$$

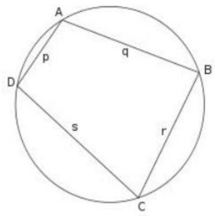
Using this identity and the fact that if (x_1, x_2) are solutions to the equations $x^2-Ny^2=k_1$ and $x^2-Ny^2=k_2$, respectively, then $(x_1x_2 + Ny_1y_2,\ x_1y_2 + x_2y_1)$ is a solution to $x^2-Ny^2=k_1k_2$. He was able to find integral solutions of the equation $x^2-Ny^2=k_i$

This result is called the "Samasa" or the principle of composition. It appears that Brahmagupta never gave proof of this lemma. The ancient Indians never paid attention to the proofs. The proof was given by Krsna (AD 1580), which has been given in this book.

Geometry:

Brahmagupta's most famous result in geometry is his formula for cyclic quadrilaterals. Given the lengths of the sides of any cyclic quadrilateral, he gave an approximate and exact formula for the figure's area. So, given the lengths p, q, r of a cyclic quadrilateral, the approx. area is $\frac{p+r}{2}.\frac{q+s}{2}$ while letting.

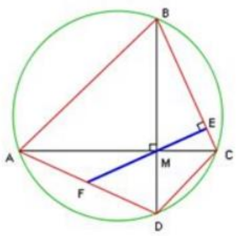
Exact Area $=\sqrt{(t-p)(t-q)(t-r)(t-s)}$.



One of his theorems on triangles states that when the base is divided by its altitude, the lengths of two segments are: $\frac{1}{2}.\,b \pm \frac{c^2-a^2}{b}$.

He further gives a theorem on rational triangles. A triangle with rational sides a, b, c and rational area is of the form

$\mathbf{a} = \frac{1}{2}\left(\frac{u^2}{v} + v\right),\quad \mathbf{b} = \frac{1}{2}\left(\frac{u^2}{w} + w\right), c = \frac{1}{2}\left(\frac{u^2}{v} - v + \frac{u^2}{w} - w\right)$ { For some rational numbers u, v, w}



In a cyclic quadrilateral whose diagonals are perpendicular to each other, the line through the point of intersection of the diagonals which is perpendicular to one side bisects the opposite side i.e., AF = FD.

He also gives formulas for the lengths and areas of geometric figures, such as the circumradius of an isosceles trapezoid and a scalene quadrilateral, and the lengths of diagonals in a scalene cyclic quadrilateral. In verse 40, he gives value for π.

In chapter twelve of his Brāhmasphuṭasiddhānta, Brahmagupta provides a formula useful for generating Pythagorean triples: Stated geometrically, this says that if a right-angled triangle has a base of length a = mx and altitude of length b = m + d, then the length, c, of its hypotenuse is given by c = m(1 + x) − d.

And, indeed, elementary algebraic manipulation shows that a2 + b2 = c2 whenever d has the value stated. Also, if m and x are rational, so are d, a, b, and c. A Pythagorean triple can therefore be obtained from a, b, and c by multiplying each of them by the least common multiple of their denominators.

Astronomy:

It was through Brāhmasphuṭasiddhānta,that the Arabs learned of Indian astronomy. In chapter 7 of his Brāhmasphuṭasiddhānta, entitled 'Lunar crescent', Brahmagupta rebuts the idea that the moon is farther from the earth than the sun, an idea which is maintained in scriptures. He does this by explaining the illumination of the moon by the sun. He explains that since the moon is closer to the earth than the sun, the degree of the illuminated part of the moon depends on the relative positions of the sun and the moon, and this can be computed from the size of the angle between the two bodies. He gave methods for calculating the position of heavenly bodies over time, their rising and setting, conjunctions, and the calculation of the solar and lunar eclipses. About the earth's gravity, he said: "Bodies fall towards the earth as it is like the earth to attract bodies, just as it is like the water to flow". A translation of Brāhmasphuṭasiddhānta, was carried out by Muhammad al-Fazari and had a far-reaching influence on subsequent Arabic works. In 860 A.D. an extensive commentary on Brāhmasphuṭasiddhānta, was written by Prthudakasvami. His work was extremely elaborate and unlike many Indian works did not 'suffer brevity' of expression.

Early concept of gravity:

Brahmagupta in 628 first described gravity as an attractive force, using the term "gurutvākarṣaṇam (गुरुत्वाकर्षणम्)" to describe it: The earth on all its sides is the same; all people on the earth stand upright, and all heavy things fall to the earth by a law of nature, for it is the nature of the earth to attract and to keep things, as it is the nature of water to flow ... If a thing wants to go deeper down than the earth, let it try. The earth is the only low thing, and seeds always return to it, in whatever direction you may throw them away, and never rise upwards from the earth.

Conclusion:

Brahmagupta, a most accomplished mathematician, lived during this medieval period and was responsible for creating good mathematics in the form of geometrical theorems and number-theoretic results. This is besides his contribution to astronomy. In estimating the mathematical work of Brahmagupta, one has to remember that he belonged to the seventh century, 14 centuries before our present time. While mathematics was described as the jewel of all sciences, Brahmagupta accordingly deserves to be described as the brightest star in the galaxy of mathematicians. Bhaskara II was greatly influenced by the work of Brahmagupta. In his admiration of Brahmagupta's talent and ingenuity, Bhaskara II showered the title, "the gem in the circle of mathematicians". Not only does Brahmagupta richly ¯ deserve the title showered on him by Bhaskara II but he also deserves a much bigger title, "the gem in the circle of mathematicians in the world history of mathematics."

References:

1. Bose, D. M.; Sen, S. N.; Subbarayappa, B. V. (1971), A Concise History of Science in India, New Delhi: Indian National Academy of Science, pp. 95–97, archived from the original on 8 December 2015.

2. Bhattacharyya, R. K. (2011), "Brahmagupta: The Ancient Indian Mathematician", in B. S. Yadav; Man Mohan (eds.), Ancient Indian Leaps into Mathematics, Springer Science & Business Media, pp. 185–192, ISBN 978-0-8176-4695-0.
3. Boyer, Carl B. (1991), A History of Mathematics, John Wiley & Sons, Inc, ISBN 0-471-54397-7.
4. Cooke, Roger (1997), The History of Mathematics: A Brief Course, Wiley-Interscience, ISBN 0-471-18082-3.
5. Gupta, Radha Charan (2008), "Brahmagupta", in Selin, Helaine (ed.), Encyclopaedia of the History of Science, Technology, and Medicine in Non-Western Cultures, Springer, pp. 162–163, ISBN 978-1-4020-4559-2