An Inventory Model for Deteriorating Items with Stock-Dependent Demand under Fuzzy Lead Time and Partial Backlogging

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Abstract:
For inventory management purposes, lead time refers to the amount of time it takes for a purchase order to be completed. Its effect is an important phenomenon in inventory management system. It also plays a significant role when lead time unknown to the decision makers. This paper deals with an inventory model for deteriorating items under fuzzy lead time. The stock dependent demand with partially backlogged shortages are considered in the proposed model. The total inventory costs for both crisp model and fuzzy model are derived. The fuzzy lead time is assumed to be triangular and trapezoidal numbers. The signed distance methods (SD) is used for defuzzification purpose. The developed model is validated with the help of numerical illustration under both crisp and fuzzy scenario. A pictorial presentation is furnished to explain the behaviour of the total inventory costs towards lead-crisp, lead-triangular and lead-trapezoidal values. Lastly a sensitivity analysis is performed to judge the sensitive behaviour of the total cost towards changes of the cost parameters.

Keywords: Inventory, fuzzy lead time, deterioration, stock dependent, partially backlogged, triangular number, trapezoidal number and defuzzification.

1. Introduction:

Most of the inventory models are developed under the assumption that the time gap between placing an order and receiving goods is negligible. This indicates that suppliers of goods are quite near retailers and ready to make the requirement quantity of goods available to the customers in a fraction of period. But practically this situation is not so fast and simple with no waste of time. In reality, there is always a time gap between placing an order and receiving items. Moreover this time gap is not exactly known because there are different types of demand patterns like increasing in time, decreasing in time, ramp type, Weibull type, linear trended in time, stock dependent etc. In this field, I mention few research papers of Chih-Hsun et al [1], Mandal N K [2], Roy A [3], Kazemi N [4], Biswajit [5], Sen N [6], Shekarian E [7], Sujit [8] etc.. This paper deals with an inventory model assuming non -zero lead time under fuzzy sense.
In real life situation, the consumption rate for many consumer goods like donuts, vegetables, cosmetics etc is sometimes influenced by stock-level. It is also observed that many customers are attracted to buy more goods when they saw a large pile of goods stuck on shelf in a supermarket. So consumption rate varying with stock level is most important part in the inventory management system. Several inventory models related to stock-dependent demand level are developed by many researchers like Biswajit [9],

Chih-Te Yang [10], Mishra U [11], Indrajit [12], Hwang H [13], Yonit [14], Biswaranjan [15] and many more.

In fact an inventory policy allows shortages with less expensive than the policy without shortages. Mishra [16], Jalan et al [17] etc developed inventory model assuming shortages which are completely backlogged. But some customers would like to wait for backlogging during the period of shortage, other would not . Consequently an opportunity cost due to lost sales should be considered in the inventory model. Neeraj et al [18], Nayek et al [19], Biswaranjan [20] and many more researchers assumed that backlogging rate was a fixed fraction of the demand rate during the period.

On the above sort of situation, the efforts have been furnished to develop an inventory model for deteriorating items under fuzzy lead period. The stock dependent demand with partially backlogged shortages are assumed in the present model. The total inventory costs for both crisp model and fuzzy model are derived. The fuzzy lead time is assumed to be triangular and trapezoidal numbers. The signed distance methods (SD) is used for defuzzification purpose. The developed model is validated with the help of numerical illustration under both crisp and fuzzy scenario. A pictorial presentation is furnished to explain the behaviour of the total inventory costs towards lead-crisp, lead-triangular and lead-trapezoidal values. Lastly a sensitivity analysis is performed to conclude the sensitive behaviour of the total cost towards changes of the cost parameters

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2. Definitions and Preliminaries:

We have stated the following definitions for development of the fuzzy inventory model.

A fuzzy set X on the given universal set is a set of order pairs and defined by

$\tilde{A} = \{(x, \lambda_{\tilde{A}}(x)) : x \in X\}$, where $\lambda_{\tilde{A}} : X \to [0,1]$ is called membership function.

a) A fuzzy number $\tilde{A}$ is a fuzzy set on the real number R, if its membership function $\lambda_{\tilde{A}}$ has the following properties

(i). $\lambda_{\tilde{A}}(x)$ is upper semi continuous.

(ii). $\lambda_{\tilde{A}}(x)$ =0, outside some interval $[a_1, a_4]$

Then ∃ real numbers a_2 and a_3 , $a_1 \le a_2 \le a_3 \le a_4$ such that $\lambda_{\tilde{A}}(x)$ is increasing on $[a_1, a_2]$ and decreasing on $[a_3, a_4]$ and $\lambda_{\tilde{A}}(x)$ =1 for each $x \in [a_2, a_3]$.

b) A triangular fuzzy number $\tilde{A} = (a_1, a_2, a_3)$ is represented with membership function $\lambda_{\tilde{A}}$ as

$$\lambda_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, a_1 \le x \le a_2; \\ \dfrac{a_3 - x}{a_3 - a_2} a_2 \le x \le a_3; \\ 0, otherwise \end{cases}$$

c) Let $\tilde{A}=(a_1,a_2,a_3)$ is a triangular fuzzy number, then the Signed Distance Method of $\tilde{A}$ is defined as $d(\tilde{A},0)= = \dfrac{a_1+2a_2+a_3}{4}$

d) A trapezoidal fuzzy number $\tilde{A}=(a_1,a_2,a_3,a_4)$ is represented with membership function $\lambda_{\tilde{A}}$ as

$$\lambda_{\tilde{A}}(x)=\begin{cases} \dfrac{x-a_1}{a_2-a_1}, a_1\le x\le a_2 \\ 1, a_2\le x\le a_3 \\ \dfrac{a_4-x}{a_4-a_3} a_3\le x\le a_4 \\ 0, otherwise \end{cases}$$

e) Let $\tilde{A}=(a_1,a_2,a_3,a_4)$ is a trapezoidal fuzzy number, then the Signed Distance Method of $\tilde{A}$ is defined as $d(\tilde{A},0)= = \dfrac{a_1+a_2+a_3+a_4}{4}$

3. Nomenclature:

1. Lead time is non-zero and fuzzy in nature
2. Replenishment rate is finite.
3. The planning horizon is infinite.
4. There is no repair of deteriorated items occurring during the cycle.
5. The demand rate is stock dependent linear trended.
6. Shortages are allowed and partially backlogged.
7. i(t) : on-hand inventory at time t.
8. T : The fixed length of order cycle.
9. I_o : Maximum inventory level in (0, T).
10. $\theta(t)=\theta$, $0<\theta<1$ is deterioration rate of an item.
11. $D(t)=\alpha+i\beta$, $\alpha,\beta>0$, α is annual demand parameter and β is demand elasticity parameter.
12. d_o : Ordering cost per order
13. d_c : Deterioration cost per unit item
14. h_c : Holding cost per unit item.
15. s_c : Shortage cost per unit item.
16. p_c : Purchase cost per unit item.
17. l_c : Lost sale cost per unit item.
18. L : lead time
19. λ : Backlogging parameter, λ>0.
20. TC : The total average cost of the system per unit time

4. Mathematical Formulation:

Crisp Model : In this environment, the time gap between placing order and receiving order is not uncertain. So as based on the above assumptions, the differential equations governing the proposed inventory model as

$$\frac{di(t)}{dt}+\theta i(t)=-(\alpha+\beta i),0\le t\le t_1 \qquad (4.1)$$

$$\frac{di(t)}{dt}=-\lambda(\alpha+\beta i),t_1\le t\le t_2 \qquad (4.2)$$

and $$\frac{di(t)}{dt}+\theta i(t)=-(\alpha+\beta i),t_2\le t\le T \qquad (4.3)$$

Boundary Conditions $i(0)=I_0$, $i(t_1)=0$ and $i(t_2)=I_0$ (4.4)

Solutions of the equations (4.1), (4.2) and (4.3) using (4.4) are the following:

$$i(t)=(I_0+\frac{\alpha}{\theta+\beta})e^{-(\theta+\beta)t}-\frac{\alpha}{\theta+\beta}, \qquad 0\le t\le t_1 \qquad (4.5)$$

$$i(t)=\frac{\alpha}{\beta}[e^{-\lambda\beta(t-t_1)}-1], \qquad t_1\le t\le t_2 \qquad (4.6)$$

And $$i(t)=(I_0+\frac{\alpha}{\theta+\beta})e^{-(\theta+\beta)(t-t_2)}-\frac{\alpha}{\theta+\beta}, \qquad t_2\le t\le T \qquad (4.7)$$

But $L=t_2-t_1$, so the equation (4.7) becomes

$$i(t)=(I_0+\frac{\alpha}{\theta+\beta})e^{-(\theta+\beta)(t-t_1-L)}-\frac{\alpha}{\theta+\beta}, \qquad t_2\le t\le T \qquad (4.8)$$

Cost Components:

The total cost (TC) has the following components

1. **Ordering Cost (OC) =** d_o
2. **Holding Cost (HC)**

$$HC=h_c[\int_0^{t_1}i(t)dt+\int_{t_1}^{T}i(t)dt]$$

$$=h_c\int_0^{t_1}[(I_0+\frac{\alpha}{\theta+\beta})e^{-(\theta+\beta)t}-\frac{\alpha}{\theta+\beta}]dt+h_c\int_0^{t_1}[(I_0+\frac{\alpha}{\theta+\beta})e^{-(\theta+\beta)(t-t_2)}-\frac{\alpha}{\theta+\beta}]dt$$

$$=h_c(I_0+\frac{\alpha}{\theta+\beta})\frac{1}{\theta+\beta}\{2-e^{-(\theta+\beta)t_1}-e^{-(\theta+\beta)(T-L-t_1)}\}-h_c\frac{\alpha}{\theta+\beta}(T-L)$$

3. **Deteriorating Cost (DC)**

$$DC=d_c[2I_0-\int_0^{t_1}(\alpha+i\beta)dt-\int_{t_2}^{T}(\alpha+i\beta)dt]$$

$$= d_c[\frac{2\theta}{\theta+\beta}I_0 + \frac{\alpha\theta}{\theta+\beta}(L-T) - \frac{2\alpha\beta}{(\theta+\beta)^2} + \frac{\beta}{\theta+\beta}(I_0 + \frac{\alpha}{\theta+\beta})(1+e^{-(\theta+\beta)(T-L)})e^{-(\theta+\beta)t_1}]$$

4. Shortage cost (SC)

$$SC = s_c\int_{t_1}^{t_2} D(t)dt = s_c\int_{t_1}^{t_2}(\alpha + i\beta)dt = s_c\{1-e^{-\lambda\beta L}\}$$

5. Purchase cost (PC)

$$PC = p_c[2I_0 - \int_{t_1}^{t_2} i(t)dt] = p_c[2I_0 - \frac{\alpha}{\beta}\int_{t_1}^{t_2}\{e^{-\lambda\beta(t-t_1)} - 1\}dt]$$

$$= p_c\frac{\alpha}{\beta}\{\frac{1}{\lambda\beta}(1-e^{-\lambda\beta L}) - L\}$$

6. Lost sale cost (LSC)

$$LSC = l_c[\int_{t_1}^{t_2} D(t)dt - \int_{t_1}^{t_2} i(t)dt] = l_c[\frac{\alpha(\beta-1)}{\lambda\beta^2}(1-e^{-\lambda\beta L}) + \frac{\alpha L}{\beta}]$$

Therefore the total cost (TC) per unit time is given by

$$\text{TC}(t_1) = \frac{1}{T}[\text{OC + HC + DC + SC + PC + LSC}]$$

$$= \frac{1}{T}[\ d_o + h_c(I_0 + \frac{\alpha}{\theta+\beta})\frac{1}{\theta+\beta}\{2-e^{-(\theta+\beta)t_1} - e^{-(\theta+\beta)(T-L-t_1)}\} - h_c\frac{\alpha}{\theta+\beta}(T-L)$$

$$+ d_c[\frac{2\theta}{\theta+\beta}I_0 + \frac{\alpha\theta}{\theta+\beta}(L-T) - \frac{2\alpha\beta}{(\theta+\beta)^2} + \frac{\beta}{\theta+\beta}(I_0 + \frac{\alpha}{\theta+\beta})(1+e^{-(\theta+\beta)(T-L)})e^{-(\theta+\beta)t_1}]$$

$$+ s_c\{1-e^{-\lambda\beta L}\} + p_c\frac{\alpha}{\beta}\{\frac{1}{\lambda\beta}(1-e^{-\lambda\beta L}) - L\} + l_c[\frac{\alpha(\beta-1)}{\lambda\beta^2}(1-e^{-\lambda\beta L}) + \frac{\alpha L}{\beta}]\ ] \qquad (4.9)$$

For minimum, the necessary condition is $\frac{dTC(t_1)}{dt_1} = 0$

Or, $$h_c\{1-e^{-(\theta+\beta)(T-L-2t_1)}\} - d_c\beta\{1+e^{-(\theta+\beta)(T-L)}\} = 0 \qquad (4.10)$$

Which is the equation for optimum solution.

Solving the equation (4.10), we get the optimum value of $t_1 = t_1^*$.

The optimum total cost of TC (t_1) is obtained from the expression (4.9) by putting $t_1 = t_1^*$.

5. Numerical Examples:

The following two numerical illustrations are given for both the inventory models namely crisp model and fuzzy model.

Example 1: (L-Crisp):

The values of the parameters be as follows

d_o = 100 per order; h_c = \$ 5 per unit ; d_c = \$ 0.75 per unit; p_c = \$ 10 per unit ; s_c = \$ 3 per unit ; l_c = \$ 8 per unit ; θ = 0.01; L = 0.34 ; λ = 0.3 ; α = 500 ; β = 2 ; I_0 = 1000 ; T = 1 year.

Solving the equation (4.10) with the help of computer using the above parameter values, we find the following optimum outputs

t_1^* = 0.329 year and TC^* = \$ 26,202.71

It is also checked that this solution satisfies the sufficient condition for optimality.

Example 2: (L-Fuzzy):

The values of the parameters be as follows

d_o = 100 per order; h_c = \$ 5 per unit ; d_c = \$ 0.75 per unit; p_c = \$ 10 per unit ; s_c = \$ 3 per unit ; l_c = \$ 8 per unit ; θ = 0.01; λ = 0.3 ; α = 500 ; β = 2 ; I_0 = 1000 ; T = 1 year.

We have the following optimum outputs

L	Triangular (0.33, 0.40, 0.47)		Trapezoidal (0.20. 0.27, 0.33, 0.47)	
	t_1^*	TC^*	t_1^*	TC^*
	0.248 year	\$ 26,658.83	0.256 year	\$ 26,122.54

6. Sensitivity analysis and Pictorial Presentation.

The table indicates the comparative study of the inventory models for the optimal total costs towards cost parameters under L-crisp, L-triangular and L-trapezoidal. Also a pictorial presentation is furnished on the basis of following data. The results are shown in the following tables.

Table A: Effect of changes in the cost parameters on the model.

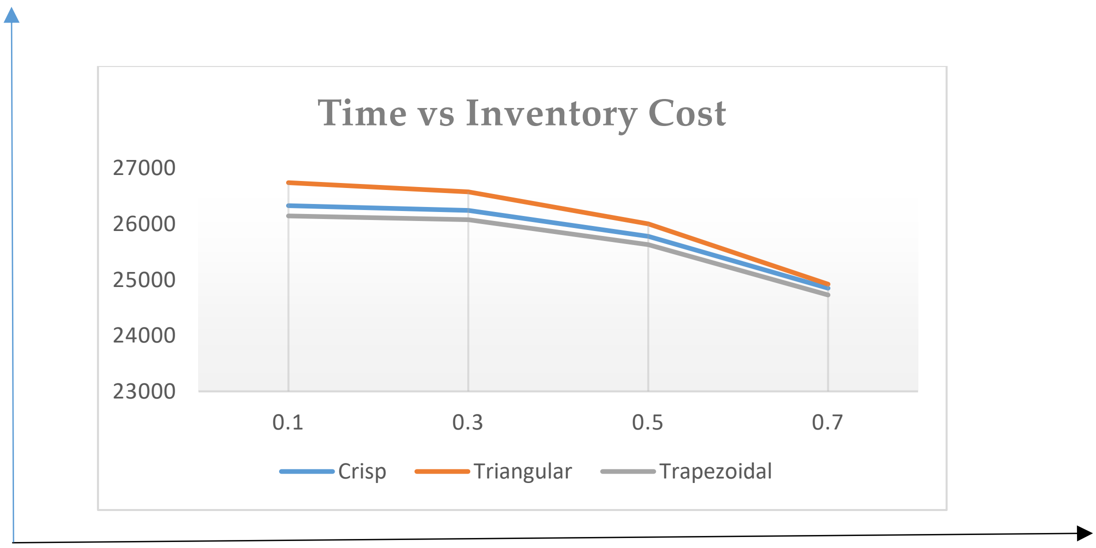
Changing parameter	% change in the system parameter	L-crisp	L-triangular	L-trapezoidal
		% change of TC^*	% change of TC^*	% change of TC^*
d_o	-50	- 0.19	-0.18	-0.19
	-20	- 0.08	-0.07	-0.08
	+20	0.08	0.07	0.08
	+50	0.19	0.18	0.19
h_c	-50	-3.89	-3.75	-4.05
	-20	-1.60	-1.51	-1.63
	+20	1.62	1.52	1.64
	+50	4.07	3.74	4.13

d_c	-50 -20 +20 +50	-0.15 -0.06 0.06 0.14	-0.03 -0.01 0.01 0.03	-0.02 -0.01 0.01 0.02
p_c	-50 -20 +20 +50	-41.65 -16.66 16.66 41.65	-41.23 -16.49 16.49 41.22	-32.11 -17.63 16.67 31.67
s_c	-50 -20 +20 +50	-0.88 -0.35 0.35 0.88	-0.99 -0.39 0.39 0.99	-0.82 -0.32 0.32 0.82
l_c	-50 -20 +20 +50	-2.47 -0.98 0.98 2.47	-2.83 -1.13 1.13 2.83	-2.31 -0.92 0.93 2.31

Table B : Optimal costs corresponding to various lead times

t_1	**Optimal values of total cost (TC) For**		
	L-crisp	L-triangular	L-trapezoidal
0.1	$ 26329.24	$ 26741.45	$ 26144.85
0.3	$ 26245.87	$ 26579.42	$ 26075.86
0.5	$ 25778.24	$ 26004.10	$ 25627.93
0.7	$ 24849.77	$ 24921.21	$ 24727.68

Inventory Cost



0 Time

Figure-1 : Pictorial presentation of Inventory Cost

From Table A, the sensitivity analysis is performed by changing all cost parameters -50%, -20%, +20% and +50% taking one cost parameter at a time and keeping the other cost parameters unchanged. The observations may be made

1. The optimum value of total cost (TC) increase or decrease with the increase or decrease in the values of the cost parameters d_o, h_c, d_c, s_c and l_c for both crisp model and fuzzy model. The results obtained show that the optimum values of total cost (TAC) are almost insensitive towards changes of the parameters d_o, h_c, d_c, s_c and l_c; whereas these are highly sensitive towards changes of the purchase cost parameter p_c.
2. From Table B, it is observed that as the time t_1 increases, the optimum values of the total cost for L-crisp, L-triangular and L-trapezoidal models decrease. Moreover, the total cost is most minimum for L-trapezoidal model amongst others.
3. The pictorial presentation (Figure 1) on inventory cost versus time shows that the value of the optimal total cost for L-trapezoidal is smaller than the values of the optimum total cost for L-crisp and L-triangular.

7. Concluding remarks:

The present article is fuzzy lead time based inventory model for deteriorating items under a stock dependent demand with partially backlogged shortages. The fuzzy lead time is assumed to be triangular and trapezoidal numbers. The signed distance methods (SD) is used for defuzzification purpose. The model is illustrated with the help of numerical examples under both crisp and fuzzy lead time scenario. A pictorial presentation is furnished to explain the behaviour of the total inventory costs towards lead-crisp, lead-triangular and lead-trapezoidal values. A sensitivity analysis of the optimal solution with pictorial presentation shows that the total inventory is highly sensitive towards changes of the purchase cost parameter p_c. It also observed that the total inventory cost for L-trapezoidal is lower than the inventory cost for L-crisp and L-triangular. This research work can be extended further to an inflationary inventory model assuming trade credit policy.

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