Approximate Solutions of Legendre's Differential Equation by Using Fractional Differential Transform Method

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Abstract
In this article, we examine the fractional order of Legendre's equations by applying the Differential Transformation Method (DTM). This method is effective and sustainable in the investigation of Legendre's equations. Series solution could be used to compare the Bessel and Legendre differential equations. If $\alpha = 1$ then, to get the solution of fractional order of Legendre's equation. Generalized fraction variation will use to calculate the fraction form of a special function. The results of the Legendre equation will compare with the exact solutions at $\alpha = 1$ and also shows that the method is quite precise and reliable.

Keywords: Legendre's Differential equation, Fractional Differential Transform Method, Exact Solution

1. Introduction

Differential equations have emerged as a significant area of study in both fundamental and applied mathematics since the middle of the seventeenth century. Even though the topic has been thoroughly studied, it is still important for research because of new connections to other mathematical fields, appropriate interactions with other fields, the intriguing evolution of basic concepts and hypotheses over time, the emergence of fresh perspectives in the 20th century, and other factors[1].
It may also be used to simulate the onset of cancer and the systemic spread of disease in medical education. It may also be applied to describe the flow of current. Economists may find it useful in developing the most effective financial plans. These equations can also be used to explain the motion of waves as well as a clock[2].
Boundary value problems arise in many areas of science and mathematics. In physics, for example, a boundary value problem can be used to model the actions of a heat-conducting rod or a vibrating string. In engineering, building a support structure or a bridge usually requires solving a boundary value problem. Boundary value problems are useful in many fields and provide a useful tool for understanding how differential equation-governed systems behave[3].
The second order ordinary differential equation is also refer to as the Legendre's differential equation. Legendre's technique are used in several fields of applied mathematics, physics, and chemistry in physical conditions involving spherical geometry, for instance, the movement of a perfect fluid around a sphere, determining the magnetic field due to a given sphere, and determining the heat distributions in a cylinder

with its exterior. Legendre's differential equation was developed by Legendre in the last years of the 18th century[4].

One mathematical strategy that generates a technique based on an electric sequence is the Differential Transform Method (DTM). This work undertakes a thorough examination of DTM and its development as a productive method for resolving a variety of mathematical issues. Like every other area of mathematics, DTM has evolved both longitudinally and laterally. Among other things, it may be used to answer fractional, partial, and typical problems in mathematics[5].

Fractional calculus may be considered because the fractional derivative extends the ordinary derivative to non-integer classifications. The equation is converted to a fractional differential equation using the derivative of fractions operator. Once the differential issue has been transformed into a fractional differential equation, it may be solved using a variety of techniques. These methods might include Laplace transforms, Mellin transforms, or other integral transforms, depending on the circumstances[6].

In this research we will find approximate solution of fractional order Legendre's differential equation by using Fractional differential transform method and compare the result obtained with the existing method.

2. Preliminaries

In this section we discuss some definitions related to our research.

Definition no 1: [7] The Legendre differential equation also known as second order ordinary differential equation is defined as

$$(1-x^2)\,\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0.$$

Definition no 2: [8]The Differential Transformation of the n^{th} the derivative of the equation f (x) at x_0 is defined as

$$F(n)=\frac{1}{n!}\left[\frac{\partial^n f(x)}{\partial x^n}\right]_{x=x_o},$$

$$f(x)=\sum_{i=0}^{\infty}\frac{f^i(x)}{i!}(x-x_0),$$

If $f(x)$ is a mathematical function on x_o, then $f(x)$ the order differential transformation is written as

$$F(x)=DT\{f(x)\},$$

$$=\left[\frac{f^k(x)}{k!}\right]_{x=x_0},$$

Definition no 3:[9] The reverse of the differential transformation is written as

$$D_T^{-1}\{F(k)\}=f(x),$$

$$=\sum_{k=0}^{\infty}F(k)(x-x_0)^k.$$

Theorem 1: Let $f(x)$ and $g(x)$ relate to differentially transformed analytical operations $F(k)$ and $G(k)$ accordingly;

Thus,

$$D_T\{\alpha f(x)+\beta g(x)\}=\alpha F(k)+\beta G(k). \qquad (a)$$

In which α and β stand for variables.

Theorem 2: Let $f(x)$ define an analytical functioning, with differential transformation $F(k)$; then,

$$D_T=\left\{\frac{d^n f(x)}{dx^n}\right\}=\frac{(k+n)!}{k!}F(k+n). \qquad (b)$$

Theorem 3: Let $f_1(x)$ and likewise $f_2(x)$ indicate analytical functions that $f(x)=f_1(x).f_2(x)$; then,

$$F(k)=\sum_{n=0}^{k}F_1(n)\,F_2(k-n). \qquad (c)$$

Theorem 4: Let $f(x)$ similar to an analytical functional with $D_T\{f(x)\}=F(k)$; thus,

$$D_T\{x^m f^n(x)\}=\sum_{i=0}^{k}\delta_{i,m}\frac{(k+n-i)!}{(k-i)!}F(k+n-i), \qquad (d)$$

And if, $m=n$, then

$$D_T\{x^m f^n(x)\}=\prod_{i=0}^{n-1}(k-i)F(k).$$

Theorem 5: Let $f(x)$ describe a mathematical function $D_T\{f(x)\}=F(k)$; then,

$$D_T\{e^{\alpha x}f^n(x)\}=\sum_{i=0}^{k}\frac{\alpha^i}{i!}\frac{(k+n-i)!}{(k-i)!}F(k+n-i),$$

$$D_T\{\cos(\alpha x)\,f^n(x)\}=\sum_{i=0}^{k}\frac{\alpha^i}{i!}\cos(\frac{i\pi}{2})\frac{(k+n-i)!}{(k-i)!}F(k+n-i),$$

And

$$D_T\{\sin(\alpha x)\,f^n(x)\}=\sum_{i=0}^{k}\frac{\alpha^i}{i!}\sin(\frac{i\pi}{2})\frac{(k+n-i)!}{(k-i)!}F(k+n-i). \qquad (e)$$

Theorem 6: Let $f(x)$ develop a function of analysis, where $D_T\{f(x)\}=F(k)$; then,

$$D_T\left\{\frac{d}{dx}(xf^n(x))\right\}=\frac{(k+1)(k+n)!}{k!}F(k+n), \qquad (f)$$

Take note of, for $n=1$, simplifies to the following formula:

$$D_T\left\{\frac{d}{dx}(xf^1(x))\right\}=(k+1)^2F(k+1).$$

Theorem 7: Let $f(x)$ represent an analytical function, whereas $D_T\{f(x)\}=F(k)$; then,

$$D_T\left\{\frac{d}{dx}(x^m f^n(x))\right\}=\frac{(k+1)(k+n-m+1)!}{(k-m+1)!}F(k+n-m+1) \qquad (g)$$

3. Solution of the Fractional Order Legendre Differential Equation

The Fractional Form of the Legendre Differential Equation.

$$(1-x^{2\alpha})D^\alpha D^\alpha y(x)-2\alpha_1 xD^\alpha y(x)+m(m+1)\alpha_1^2 y(x)=0\,,$$

$$D^\alpha y(x)=\frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha+1)}x^{1-\alpha}\frac{dy}{dx},$$

$$D^\alpha D^\alpha y(x)=\frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha+1)}D^\alpha(x^{1-\alpha}\frac{dy}{dx})\,,$$

$$D^\alpha D^\alpha y(x)=\left[\frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha+1)}\right]^2\left[x^{2(1-\alpha)}\frac{d^2y}{d^2x}+x^{2(1-\alpha)}\frac{dy}{dx}\right],$$

$$D^\alpha(x^{1-\alpha}\frac{dy}{dx})=D^\alpha\left(x^{(1-\alpha)}\right)\frac{dy}{dx}+x^{1-\alpha}D^\alpha\frac{dy}{dx},$$

$$= \left(\frac{\Gamma(\gamma)}{\Gamma(\gamma - \alpha + 1)} x^{(1-2\alpha)} \frac{dy}{dx} + x^{1-\alpha} \frac{\Gamma(\gamma)}{\Gamma(\gamma - \alpha + 1)} x^{1-\alpha} D^{\alpha} y(x) \right) \times$$
$$\left(\frac{\Gamma(\gamma)}{\Gamma(\gamma - \alpha + 1)} \left[x^{(1-2\alpha)} \frac{dy}{dx} + x^{2(1-\alpha)} \frac{d^2y}{d^2x} \right] \right),$$

Where,

$$\alpha_1 = \frac{\Gamma(\gamma)}{\Gamma(\gamma - \alpha + 1)} \alpha \,,$$
$$(1 - x^{2\alpha}) D^{\alpha} D^{\alpha} y(x) - 2\alpha_1 x D^{\alpha} y(x) + m(m+1)\alpha_1^2 y(x) = 0 \,.$$
$$\mathrm{y} = \sum_{n=0}^{\infty} C_n x^{\alpha n} \,,$$
$$D^{\alpha} y(x) = \frac{\Gamma(\gamma)}{\Gamma(\gamma - \alpha + 1)} \sum_{n=1}^{\infty} \alpha n C_n x^{\alpha n - \alpha} \,,$$
$$D^{\alpha} y(x) = \frac{\Gamma(\gamma)}{\Gamma(\gamma - \alpha + 1)} \alpha \sum_{n=1}^{\infty} n C_n x^{\alpha(n-1)} \,,$$
$$D^{\alpha} D^{\alpha} y(x) = \left[\frac{\Gamma(\gamma)}{\Gamma(\gamma - \alpha + 1)} \right]^2 \alpha^2 \sum_{n=2}^{\infty} n(n-1) C_n x^{\alpha(n-2)},$$
$$\alpha_1 = \frac{\Gamma(\gamma)}{\Gamma(\gamma - \alpha + 1)} \alpha \,,$$
$$(1 - x^{2\alpha})(\alpha_1)^2 \sum_{n=2}^{\infty} n(n-1) C_n x^{\alpha(n-2)} - 2x(\alpha_1)^2 \sum_{n=1}^{\infty} n C_n x^{\alpha(n-1)} + m(m+1)(\alpha_1)^2 \sum_{n=0}^{\infty} C_n x^{\alpha n} = 0 \,,$$
$$(\alpha_1)^2 \sum_{n=2}^{\infty} n(n-1) C_n x^{\alpha(n-2)} - (\alpha_1)^2 \sum_{n=2}^{\infty} n(n-1) C_n x^{\alpha n} - 2(\alpha_1)^2 \sum_{n=1}^{\infty} n C_n x^{\alpha n}$$
$$+ m(m+1)\,(\alpha_1)^2 \sum_{n=0}^{\infty} C_n x^{\alpha n} = 0 \,,$$
$$(\alpha_1)^2 \left[\sum_{n=2}^{\infty} n(n-1) C_n x^{\alpha(n-2)} - \sum_{n=2}^{\infty} n(n-1) C_n x^{\alpha n}\, 2 \sum_{n=1}^{\infty} n C_n x^{\alpha n} + m(m+1) \sum_{n=0}^{\infty} C_n x^{\alpha n} \right] = 0 \,,$$

$\alpha_1 \neq 0$, then

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{\alpha(n-2)} - \sum_{n=2}^{\infty} n(n-1) C_n x^{\alpha n} - 2 \sum_{n=1}^{\infty} n C_n x^{\alpha n} + m(m+1) \sum_{n=0}^{\infty} C_n x^{\alpha n} = 0 \,,$$

Put, $n = k + 2,\qquad n = k,$

$$\sum_{k=0}^{\infty} (k+2)(k+1) C_{k+2} x^{\alpha k} - \sum_{k=2}^{\infty} k(k-1) C_k x^{\alpha k} - 2 \sum_{k=1}^{\infty} k C_k x^{\alpha k} + m(m+1) \sum_{k=0}^{\infty} C_k x^{\alpha k}$$
$$= 0 \,,$$

$$2(1)C_2x^0 + 3(2)C_3x^\alpha + \sum_{k=2}^{\infty}(k+2)(k+1)C_{k+2}x^{\alpha k} - \sum_{k=2}^{\infty}k(k-1)C_kx^{\alpha k} - 2(1)\,C_1x^\alpha$$
$$-2\sum_{k=2}^{\infty}kC_kx^\alpha + m(m+1)C_0x^0 + m(m+1)C_1x^\alpha + m(m+1)\sum_{k=2}^{\infty}C_kx^{\alpha k} = 0\,,$$
$$[2C_2 + m(m+1)C_0]x^{\alpha(0)} + [6C_3 - 2C_1 + m(m+1)C_1]x^\alpha + \left[\sum_{k=2}^{\infty}\{(k+2)(k+1)C_{k+2} - k(k-1)C_k - 2kC_k + m(m+1)C_k\}\right]x^{\alpha k} = 0\,,$$

Since, Series is identically zero.

So,

$$C_2 = \frac{-m(m+1)}{2!}C_0\,, \tag{1}$$

$$C_3 = -\frac{(m+2)(m-1)}{3!}C_1 \tag{2}$$

Hence

$$C_{k+2} = -\frac{(m-k)(k+m+1)}{(k+2)(k+1)}C_k \tag{3}$$

By putting, $k = 2$ in equation (3) we get

$$C_4 = \frac{m(m-2)(m+1)(m+3)}{4!}C_0 \tag{4}$$

By putting, $k = 3$ in equation (3) we get

$$C_5 = \frac{(m-1)(m-3)(m+2)(m+4)}{3!}C_1 \tag{5}$$

By putting, $k = 4$ in equation (3) we get

$$C_6 = -\frac{(m-2)(m-4)m(m+1)(m+3)(m+5)}{6!}C_0 \tag{6}$$

By putting, $k = 5$ in equation (3) we get

$$C_7 = -\frac{(m-1)(m-3)(m-5)(m+2)(m+4)(m+6)}{7!}C_1 \tag{7}$$

And so on…

$$\mathrm{y} = \sum_{n=0}^{\infty}C_nx^{\alpha n}\,,$$

$$y = C_0 + C_1x^\alpha + C_2x^{2\alpha} + C_3x^{3\alpha} + C_4x^{4\alpha} + C_5x^{5\alpha} + C_6x^{6\alpha} + C_7x^{7\alpha}\,\ldots, \tag{8}$$

If we choose $y(1) = C_1\,,\; y(0) = C_0$

$C_0 = 1$ And $C_1 = 0\,.$

Put $m = 4\,, C_0 = 1\,,$

$$C_2 = \frac{-4(4+1)}{2},$$
$$C_2 = -10\,.$$

Put $m = 4\,, C_1 = 0\,,$ in equation (3)

$$C_3 = \frac{2-4(4+1)}{6}(0)\,,$$

$$C_3 = 0\,.$$

Put $m = 4$, $C_0 = 1$, in equation (4)

$$C_4 = \frac{4(4-2)(4+1)(4+3)}{4},$$

$$C_4 = \frac{35}{3}.$$

Put $m = 4$, $C_1 = 0$, in equation (5)

$$C_5 = 0\,.$$

Put $m = 4$, $C_0 = 1$, in equation (6)

$$C_6 = -\frac{(4-2)(4-4)4(4+1)(4+3)(4+5)}{6!},$$

$$C_6 = 0\,.$$

Put all the above values in Eq. (1)

$$y = 1 + (0)x^{\alpha} + (-10)x^{2\alpha} + (0)x^{3\alpha} + \frac{35}{3}x^{4\alpha} + (0)x^{5\alpha} + (0)x^{6\alpha} + (0)x^{7\alpha}$$

$$y_1(x)^{\alpha} = 1 - 10x^{2\alpha} + \frac{35}{3}x^{4\alpha}.$$

If we choose $y(1) = C_0$, $y(0) = C_1$

$C_0 = 0$ And $C_1 = 1$.

Put $m = 4$, $C_0 = 0$, then

$$C_2 = 0\,.$$

$$C_3 = \frac{2 - m(m+1)}{6}C_1,$$

Put $m = 4$, $C_1 = 1$, in equation (3)

$$C_3 = \frac{2 - 4(4+1)}{6},$$

$$C_3 = -3\,.$$

Put $m = 4$, $C_0 = 0$, in equation (4)

$$C_4 = \frac{4(4-2)(4+1)(4+3)}{4!}(0),$$

$$C_4 = 0\,.$$

Put $m = 4$, $C_1 = 1$, in equation (5)

$$C_5 = \frac{(4-1)(4-3)(4+2)(4+4)}{3.2.1},$$

$$C_5 = 24\,.$$

Put $m = 4$, $C_0 = 0$, in equation (6)

$$C_6 = -\frac{(4-2)(4-4)4(4+1)(4+3)(4+5)}{6!}(0),$$

$$C_6 = 0\,.$$

Put $m = 4$, $C_1 = 1$, in equation (7)

$$C_7 = -\frac{(4-1)(4-3)(4-5)(4+2)(4+4)(4+6)}{7.6.5.4.3.2.1},$$

$$C_7 = \frac{2}{7}.$$

Put all the above values in Eq. (8)

$$y = 0 + x^{\alpha} + 0x^{2\alpha} + (-3)x^{3\alpha} + 0x^{4\alpha} + 24x^{5\alpha} + 0x^{6\alpha} + \frac{2}{7}x^{7\alpha}$$

$$y_2(x)^{\alpha} = x^{\alpha} - 3x^{3\alpha} + 24x^{5\alpha} + \frac{2}{7}x^{7\alpha}.$$

The exact equation of Fractional Order Legendre Differential Equation.

$$y(x) = y_1(x)^{\alpha} + y_2(x)^{\alpha},$$

$$y(x) = 1 - 10x^{2\alpha} + \frac{35}{3}x^{4\alpha} + x^{\alpha} - 3x^{3\alpha} + 24x^{5\alpha} + \frac{2}{7}x^{7\alpha},$$

$$y(x) = 1 + x^{\alpha} - 10x^{2\alpha} - 3x^{3\alpha} + \frac{35}{3}x^{4\alpha} + 24x^{5\alpha} + \frac{2}{7}x^{7\alpha}.$$

From Eq. (1)

$$y = C_0 + C_1x^{\alpha} + C_2x^{2\alpha} + \cdots,$$

$$y(x) = C_0y_1(x)^{\alpha} + C_1y_2(x)^{\alpha},$$

$$y_1(x) = \left[C_0 - \frac{m(m+1)}{2!}C_0x^{2\alpha} + \frac{m(m-2)(m+1)(m+3)}{4!}C_0x^{4\alpha} - \frac{(m-4)(m-2)m(m+1)(m+3)(m+5)}{6!}C_0x^{6\alpha} + \cdots\right],$$

$$y_1(x) = C_0\left[1 - \frac{m(m+1)}{2!}x^{2\alpha} + \frac{m(m-2)(m+1)(m+3)}{4!}x^{4\alpha} - \frac{(m-4)(m-2)m(m+1)(m+3)(m+5)}{6!}x^{6\alpha} + \cdots\right],$$

$$y_2(x) = \left[C_1x^{\alpha} - \frac{(m-1)(m+2)}{3!}C_1x^{3\alpha} + \frac{(m-3)(m-1)(m+2)(m+4)}{5!}C_1x^{5\alpha} - \frac{(m-5)(m-3)(m-1)(m+2)(m+4)(m+6)}{7!}C_1x^{7\alpha} + \cdots\right].$$

Note that, if m is an even integer, the1st series vanishes meanwhile$y_2(x)$ denotes an infinite series.

For $m = 4$, then

$$y_1(x) = C_0\left[1 - \frac{m(m+1)}{2!}x^{2\alpha} + \frac{m(m-2)(m+1)(m+3)}{4!}x^{4\alpha} - \frac{(m-4)(m-2)m(m+1)(m+3)(m+5)}{6!}x^{6\alpha} + \cdots\right],$$

$$y_1(x) = C_0\left[1 - \frac{4(4+1)}{2!}x^{2\alpha} + \frac{4(4-2)(4+1)(4+3)}{4!}x^{4\alpha} - \frac{(4-4)(4-2)4(4+1)(4+3)(4+5)}{6!}x^{6\alpha} + \cdots\right],$$

$$y_1(x) = C_0\left[1 - \frac{4(4+1)}{2!}x^{2\alpha} + \frac{4(4-2)(4+1)(4+3)}{4!}x^{4\alpha} - 0\right],$$

$$y_1(x) = C_0\left[1 - \frac{4(5)}{2.1}x^{2\alpha} + \frac{4(2)(5)(7)}{4.3.2.1}x^{4\alpha}\right],$$

$$y_1(x) = C_0\left[1 - 10x^{2\alpha} + \frac{35}{3}x^{4\alpha}\right]$$

The series m contains odd integers. To put it a different way, we obtain nth degree exponential when m is a non-negative integer. The result is a fixed number of the Legendre equation. Note that it is a specific value for C_0 and C_1 which depend on whether m denotes an odd or even positive integer.
For, $m = 0$ we select $C_0 = 0$,
For, $m = 2,4,6$
Put, $m = 0$

$$C_0 = (-1)^{\frac{n}{2}} \frac{1.3 \ldots (n-1)}{2.4 \ldots n},$$

Where $m = 1$ and $C_1 = 1$

$$C_1 = (-1)^{\frac{n-1}{2}} \frac{1.3 \ldots n}{2.4 \ldots (n-1)}.$$

For example, $m = 4$

$$y_1(x) = (-1)^{\frac{4}{2}} \frac{1.3}{2.4}\left[1 - 10x^{2\alpha} + \frac{35}{3}x^{4\alpha}\right],$$

$$y_1(x) = \frac{1}{8}[3 - 30x^{2\alpha} + 35x^{4\alpha}].$$

	x	F=Exact	$Z = FDTM$	|F-Z|
$\alpha = 1$	0.0	3.00000569	3.00000000	0.00000001
	0.1	2.96840667	2.96293750	0.00546917
	0.2	2.80234667	2.85700012	0.05465345
	0.3	2.56172503	2.59567523	0.03261175
	0.4	2.53242856	2.51270000	0.01972856
	0.5	2.36416667	2.33593750	0.03177083
	0.6	2.19024000	2.21700000	0.02676000
	0.7	2.17584667	2.21293750	0.04290917
	0.8	2.10698667	2.17200000	0.04149867
	0.9	2.13926000	2.11293750	0.02163224
	1.0	3.66536235	3.62500000	0.04036235

Table 4.1: Comparison of Exact solution with $FDTM$ at $\alpha = 1$.

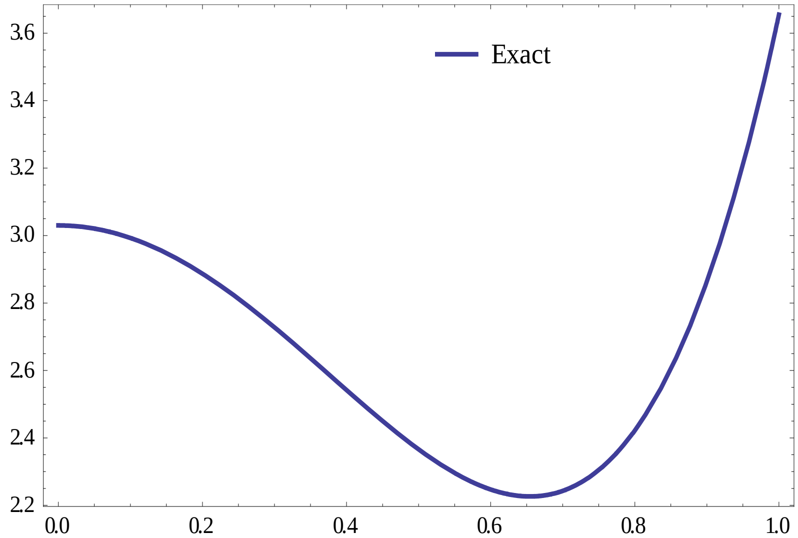

Figure 1: Graph of Exact Solution of Legendre Differential Equation

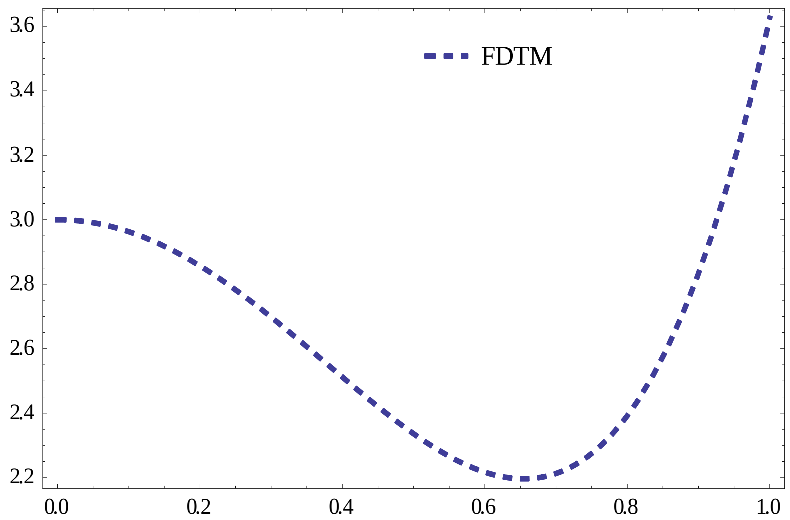

Figure 2: Graph of Approximate Solution of FDTM at $\alpha = 1$.

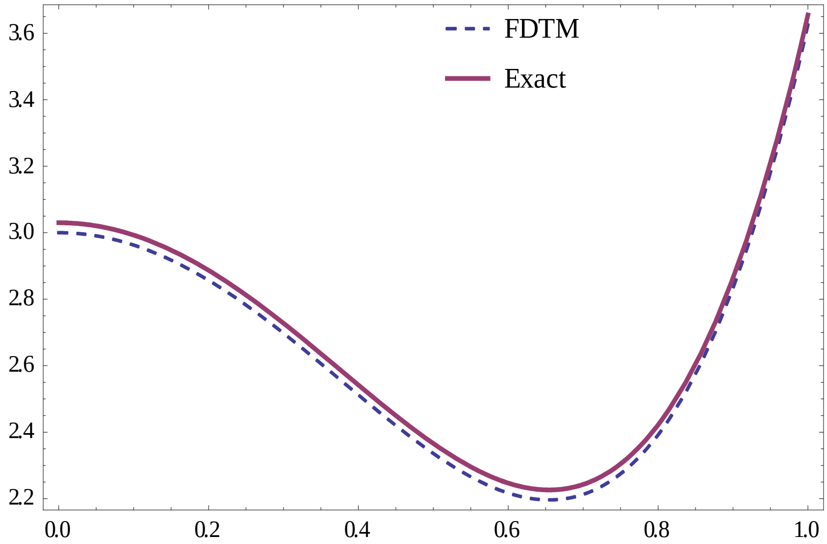

Figure 3: Comparison of Exact and FDTM Solutions at $\alpha = 1$.

4. Conclusion

In this work, the Differential Transformation Method (DTM) has shown to be an effective and trustworthy technique for deciphering Legendre's equations' fractional order. Through the use of this technique, we have been able to get series solutions that enable a thorough comparison of the Bessel and Legendre differential equations. The precision of the DTM is shown by our results, which show a high degree of accuracy and alignment with the precise solutions, especially at $\alpha = 1$. Additionally, new directions for study and application are made possible by the use of generalized fraction variation in determining the fractional form of special functions. All things considered, the results show that DTM is both practical and sustainable for the continued investigation of Legendre's equations, offering a strong basis for further research in this area.

5. References

1. P. Deuflhard and F. Bornemann, *Scientific computing with ordinary differential equations*, vol. 42. Springer Science & Business Media, 2012.
2. T. Archibald, C. Fraser, and I. Grattan-Guinness, "The history of differential equations, 1670–1950," *Oberwolfach reports*, vol. 1, no. 4, pp. 2729–2794, 2005.
3. J. C. Robinson and C. Pierre, "Infinite-dimensional dynamical systems: An introduction to dissipative parabolic pdes and the theory of global attractors. cambridge texts in applied mathematics," *Appl. Mech. Rev.*, vol. 56, no. 4, pp. B54–B55, 2003.
4. J. Billingham, A. C. King, and S. R. Otto, "Differential Equations. Linear, Nonlinear, Ordinary, Partial." Cambridge Univ. Press, Cambridge, 2003.
5. A. A. Kilbas and J. J. Trujillo, "Differential equations of fractional order: methods results and problem—I," *Appl. Anal.*, vol. 78, no. 1–2, pp. 153–192, 2001.
6. Z. M. Odibat, C. Bertelle, M. A. Aziz-Alaoui, and G. H. E. Duchamp, "A multi-step differential transform method and application to non-chaotic or chaotic systems," *Comput. Math. with Appl.*, vol. 59, no. 4, pp. 1462–1472, 2010.
7. W. N. Everitt, L. L. Littlejohn, and R. Wellman, "Legendre polynomials, Legendre–Stirling numbers, and the left-definite spectral analysis of the Legendre differential expression," *J. Comput. Appl. Math.*, vol. 148, no. 1, pp. 213–238, 2002.
8. I. H. A.-H. Hassan, "Application to differential transformation method for solving systems of differential equations," *Appl. Math. Model.*, vol. 32, no. 12, pp. 2552–2559, 2008.
9. Q. Tian, F. Yin, T. Sakaguchi, and K. Nagai, "Reverse transformation behavior of a prestrained MnCu alloy," *Acta Mater.*, vol. 54, no. 7, pp. 1805–1813, 2006.