International Journal for Multidisciplinary Research (IJFMR)
E-ISSN: 2582-2160 ● Website: www.ijfmr.com ● Email: editor@ijfmr.com

On Formulating Sequences of Diophantine 3-Tuples Through Matrix Multiplication

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Abstract
This paper illustrates the process of obtaining sequences of Diophantine 3-tuples with suitable property through matrix multiplication in two sections A and B.

Keywords: Diophantine 3-tuple, Sequence of Diophantine 3-tuples, Matrix multiplication

1. INTRODUCTION

The essence of mathematical calculations is represented by numbers and they exhibit fascinating and beautiful varieties of patterns, namely, polygonal numbers, Fibonacci numbers, Lucas numbers, Ramanujan numbers, Kynea numbers, Jacobsthal numbers and so on. In this paper, a pattern of numbers known as Diophantine 3-tuple is considered. A set of three distinct integers is called Diophantine 3-tuple with property D(n) if the product of any two members of the set with the addition of n (a non-zero integer or a polynomial with integer coefficients) is a perfect square. One may refer [1-13] for an extensive review of various problems on Diophantine triples with suitable properties.

This paper has two sections A and B. Section A illustrates the process of obtaining sequences of Diophantine 3-tuples with property $D\left(n^2a^2+2(n-1)a\right)$ through matrix multiplication and in Section B with property $D(n^2a^2+2(n-1)a+1)$.

Method of Analysis:

Section A : Diophantine 3-tuple with property $D(n^2a^2+2(n-1)a)$

Initially, construct a diophantine 2-tuple with property $D\left(n^2a^2+2(n-1)a\right)$ and then,

extend it to diophantine 3-tuple.

Let $1, 2a+1$ be two distinct integers such that

$1*(2a+1)+n^2a^2+2(n-1)a=(na+1)^2$, a perfect square

Therefore, the pair $(1, 2a+1)$ represents diophantine 2-tuple with the property $D(n^2a^2+2(n-1)a)$.

If c is the 3rd tuple,then it satisfies the following system of double equations

$$c+n^2a^2+2(n-1)a=p^2 \quad (1)$$

$$(2a+1)c+n^2a^2+2(n-1)a=q^2 \quad (2)$$

Eliminating c between (1) and (2), we have

$$(2a+1)p^2-q^2=2a\left(n^2a^2+2(n-1)a\right) \quad (3)$$

Taking

$$p=X+T \ , \ q=X+(2a+1)T \quad (4)$$

in (3) and simplifying we get

$$X^2=(2a+1)T^2+a^2n^2+2(n-1)a$$

which is satisfied by $T=1$, $X=na+1$

In view of (4) and (1), it is seen that

$$c=2a(n+1)+4$$

Note that $(1,2a+1,2a(n+1)+4)$ represents diophantine 3-tuple with property $D(n^2a^2+2(n-1)a)$.

The process of obtaining sequences of diophantine 3-tuples with property $D\left(n^2a^2+2(n-1)a\right)$ is illustrated below:

Let M be a 3×3 square matrix given by

$$M=\begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} \quad (5)$$

Now,

$$(1,2a+1,2a(n+1)+4)M=(1,2a(n+1)+4,4an+2a+9)$$

Note that

$$1*(2a(n+1)+4)+n^2a^2+2(n-1)a=(na+2)^2$$

$$1*(4an+2a+9)+n^2a^2+2(n-1)a=(na+3)^2$$

$$(2an+2a+4)*(4an+2a+9)+n^2a^2+2(n-1)a=(3an+2a+6)^2$$

$\therefore$ The triple $(1,2a(n+1)+4,4an+2a+9)$ represents diophantine 3-tuple with property $D\left(n^2a^2+2(n-1)a\right)$.

The repetition of the above process leads to sequences of diophantine 3-tuples whose general form $(1,c_{s-1},c_s)$ is given by

$$\left(1,\ s^2+2ans+2a-2an\ ,\ s^2+2ans+2s+2a+1\right),s=1,2,3.......$$

A few diophantine 3-tuples with property $D\left(n^2a^2+2(n-1)a\right)$ are given below:

$$(1, c_0, c_1) = (1, 2a+1, 2a(n+1)+4),$$
$$(1, c_1, c_2) = (1, 2a(n+1)+4, 4an+2a+9),$$
$$(1, c_2, c_3) = (1, 4an+2a+9, 6an+2a+16),$$
$$(1, c_3, c_4) = (1, 6an+2a+16, 8an+2a+25),$$
$$(1, c_4, c_5) = (1, 8an+2a+25, 10an+2a+36), \ldots$$

It is noted that the triple $(c_{s-1}, c_s+1, c_{s+1})$, $s = 1,2,3\ldots\ldots$ forms an arithmetic progression.

Remark:

Instead of (5), suppose we have a third order square matrix N given by

$$N = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}$$

Following the procedure presented above, one obtains sequences of Diophantine 3-tuples represented by $(1, 2a+1, 2a(n+1)+4)$, $(2a+1, 2a(n+1)+4, 4an+8a+9)$, $(2a(n+1)+4, 4an+8a+9, 12an+18a+25)$, $(4an+8a+9, 12an+18a+25, 30an+50a+64)$ and so on, each with property $D(n^2a^2+2(n-1)a)$.

Section B: Diophantine 3-tuple with property $D(n^2a^2+2(n-1)a+1)$

Initially, construct a diophantine 2-tuple with property $D(n^2a^2+2(n-1)a+1)$ and then,

extend it to diophantine 3-tuple.

Let $1, 2a$ be two distinct integers such that

$$1*(2a) + n^2a^2 + 2(n-1)a + 1 = (na+1)^2, \text{ a perfect square}$$

Therefore, the pair $(1, 2a)$ represents diophantine 2-tuple with the property $D(n^2a^2+2(n-1)a+1)$.

If c is the 3rd tuple,then it satisfies the following system of double equations

$$c + n^2a^2 + 2(n-1)a + 1 = p^2 \quad (6)$$

$$(2a)c + n^2a^2 + 2(n-1)a + 1 = q^2 \quad (7)$$

Eliminating c between (6) and (7), we have

$$(2a)p^2 - q^2 = (2a-1)(n^2a^2 + 2(n-1)a + 1) \quad (8)$$

Taking

$$p = X + T, \quad q = X + (2a)T \quad (9)$$

in (8) and simplifying we get

$$X^2 = (2a)T^2 + a^2n^2 + 2(n-1)a + 1$$

which is satisfied by $T = 1$, $X = na + 1$

In view of (9) and (6), it is seen that

$$c = 2a(n+1)+3$$

Note that $(1, 2a, 2a(n+1)+3)$ represents diophantine 3-tuple with property $D(n^2a^2+2(n-1)a+1)$.

The process of obtaining sequences of diophantine 3-tuples with property $D(n^2a^2+2(n-1)a+1)$ is illustrated below:

Now

$$(1, 2a, 2a(n+1)+3)M = (1, 2a(n+1)+3, 4an+2a+8)$$

Note that

$$1*(2a(n+1)+3)+n^2a^2+2(n-1)a+1 = (na+2)^2$$

$$1*(4an+2a+8)+n^2a^2+2(n-1)a+1 = (na+3)^2$$

$$(2an+2a+3)*(4an+2a+8)+n^2a^2+2(n-1)a+1 = (3an+2a+5)^2$$

$\therefore$ The triple $(1, 2a(n+1)+3, 4an+2a+8)$ represents diophantine 3-tuple with property $D(n^2a^2+2(n-1)a+1)$.

The repetition of the above process leads to sequences of diophantine 3-tuples whose general form $(1, c_{s-1}, c_s)$ is given by

$$(1,\ s^2+2ans+2a-2an-1,\ s^2+2ans+2s+2a),\ s = 1,2,3.......$$

A few diophantine 3-tuples with property $D(n^2a^2+2(n-1)a+1)$ are given below:

$$(1, c_0, c_1) = (1, 2a, 2a(n+1)+3),$$

$$(1, c_1, c_2) = (1, 2a(n+1)+3, 4an+2a+8),$$

$$(1, c_2, c_3) = (1, 4an+2a+8, 6an+2a+15),$$

$$(1, c_3, c_4) = (1, 6an+2a+15, 8an+2a+24),$$

$$(1, c_4, c_5) = (1, 8an+2a+24, 10an+2a+35), \ldots$$

It is noted that the triple $(c_{s-1}, c_s+1, c_{s+1})$, $s = 1,2,3.......$ forms an arithmetic progression.

Remark:

Considering the third order square matrix N given in Section A and following the procedure presented above, one obtains sequences of Diophantine 3-tuples represented by $(1, 2a, 2a(n+1)+3)$, $(2a, 2a(n+1)+3, 4an+8a+5)$, $(2a(n+1)+3, 4an+8a+5, 12an+18a+16)$, $(4an+8a+5, 12an+18a+16, 30an+50a+39)$ and so on, each with property $D(n^2a^2+2(n-1)a+1)$.

Note :

Taking $a = 9$ and replacing $9n$ by $k+3$, the results presented in [13] are obtained.

Conclusion

In this paper, an elegant way of obtaining sequences of Diophantine 3-tuples with suitable property has been illustrated through employing matrix multiplication. To conclude, one may search for other choices of Matrices for the formulation of sequences of Diophantine 3-tuples with suitable properties.

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