Weak Contraction for Coupled Fixed Point using Q − Function

Rajesh Shrivastava[1], Saching Mokhle[2]

[1]Research Supervisor, Department of Mathematics, Govt. Dr. Shyama Prashad Mukherjee, Science & Commerce College, (Old Benajir) Bhopal (M.P.). INDIA
[2]Research Scholar, Department of Mathematics, Govt. Dr. Shyama Prashad Mukherjee, Science & Commerce College, (Old Benajir) Bhopal (M.P.). INDIA

Abstract:
The purpose of this article is to prove coupled fixed point theorem for non linear contractive mappings in partially ordered complete quasi - metric spaces using the concept of monotone mapping with a $Q -$ function q and $(\alpha - \Psi) -$ contractive condition. The presented theorems are generalization and extension of the recent coupled fixed point theorems due to Bhaskar and Lakshmikantham \cite(BL). We also give an example in support of our theorem.

Keywords: Coupled fixed point, Coupled Coincidence point, Mixed monotone mapping.

Introduction and Preliminaries

We start, if $(X, \leq)$ is a partially ordered set and $F : X \to X$ such that for each $x, y \in X, x \leq y$ implies $F(x) \leq F(y)$, then a mapping F is said to be non decreasing. Similarly, a non increasing mapping is defined. Bhaskar and Lakshmikantham [9] introduced the following notions of a mixed monotone mapping and a coupled fixed point.

Definition 1: Let $(X, \leq)$ is a partially ordered set and $F : X \times X \to X$. The mapping F is said to have the mixed monotone property if F is nondecreasing monotone in first argument and is a nonincreasing monotone in its second argument, that is, for any $x, y \in X$

$$x_1, x_2 \in X,\ x_1 \leq x_2 \to F(x_1, y) \leq F(x_2, y)$$
$$y_1, y_2 \in X,\ y_1 \leq y_2 \to F(x, y_1) \geq F(x, y_2)$$

Definition 2: An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F: X \times X \to X$ if

$$F(x, y) = x,\ F(y, x) = y.$$

Definition 3: Let X be a nonempty set. A real valued function $d: X \times X \to R^+$ is said to be quasi metric space on X if

$[(M_1)]$ $d(x, y) \geq 0$ for all $x, y \in X$,
$[(M_2)]$ $d(x, y) = 0$ if and only if $x = y$,
$[(M_3)]$ $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

The pair (X,d) is called a quasi- metric space.

Definition 4: Let (X,d) be a quasi metric space. A mapping $q : X \times X \to R^+$ is called a Q- function on X if the following conditions are satisfied:

$[(\mathbf{Q_1})]$ for all $x, y, z \in X$,

[($\mathbf{Q_2}$)]**if** $x \in X$ and $(y_n)_{n \geq 1}$ is a sequence in X such that it converges to point y (with respect to quasi metric) and $q(x, y_n) \leq M$ for some $M = M(x)$, then $q(x, y) \leq M$;

[($\mathbf{Q_3}$)]**for** any $\epsilon > 0$ there exists $\delta > 0$ such that $q(z, x) \leq \delta$ and $q(z, y) \leq \delta$ implies that $d(x, y) \leq \epsilon$.

Remark 5: If (X, d) is a metric space, and in addition to $(Q_1) - (Q_3)$, the following condition are also satisfied:

[($\mathbf{Q_4}$)] for any sequence $(x_n)_{n \geq 1}$ in X with $\lim_{n \to \infty} \sup \{ q(x_n, x_m) : m > n \} = 0$ and if there exist a sequence $(y_n)_{n \geq 1}$ in X such that $\lim_{n \to \infty} q(x_n, y_n) = 0$, then $\lim_{n \to \infty} d(x_n, y_n) = 0$.

Then a Q- function is called $\tau -$ function, introduced by Lin and Du [16]also in the same paper [16] they show that every $\omega -$ function, introduced and studied by Kada et al. [15], is a $\tau -$ function. In fact, if we consider (X, d) as a metric space and replace (Q_2) by the following condition:

[($\mathbf{Q_5}$)] for any $x \in X$, the function $p(x, .) \to R^+$ is lower semi continuous,

then a Q- function is called a $\omega -$ function on X. Several examples of $\omega -$ functions are given in [15]. It is easy to see that if $(q(x, .))$ is lower semi continuous, then (Q_2) holds. Hence, it is obvious that every $\omega -$ function is $\tau -$ function and every $\tau -$ function is Q- function, but the converse assertions do not hold.

Example 6: Let X = R. Define $d: X \times X \to R^+$ by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ |y| & \text{otherwise} \end{cases}$$

and $q : X \times X \to R^+$ by

$$q(x, y) = | y |, \quad \forall x, y \in X.$$

Then one can easily see that d is a quasi- metric space and q is a Q- function on X, but q is neither a $\tau -$ function nor a $\omega -$ function.

Example 7: Define $d: X \times X \to R^+$ by

$$d(x, y) = \begin{cases} y - x & \text{if } x = y \\ 2(x - y) & \text{otherwise} \end{cases}$$

and $q : X \times X \to R^+$ by

$$q(x, y) = | x - y |, \ \forall x, y \in X.$$

Then one can easily see that d is a quasi- metric space and q is a Q- function on X, but q is neither a $\tau -$ function nor a $\omega -$ function, because (X, d) is not a metric space.

The following lemma lists some properties of a Q- function on X which are similar to that of a $\omega -$ function (see [15]).

Lemma 8: Let $q : X \times X \to R^+$ be a Q - function on X. Let $\{ x_n \}_{n \in N}$ and $\{ y_n \}_{n \in N}$ be sequences in X, and let $\{ \alpha_n \}_{n \in N}$ and $\{ \beta_n \}_{n \in N}$ be such that they converges to 0 and $x, y, z \in X$. Then, the following hold:

i. if $q(x_n, y) \leq \alpha_n$ and $q(x_n, z) \leq \beta_n$ for all $n \in N$, then $y = z$. In particular, if $q(x, y) = 0$ and $q(x, z) = 0$ then $y = z$;
ii. if $q(x_n, y_n) \leq \alpha_n$ and $q(x_n, z) \leq \beta_n$ for all $x \in N$, then $\{ y_n \}_{n \in N}$ converges to z;
iii. if $q(x_n, x_m) \leq \alpha_n$ for all $n, m \in N$ with $m > n$, then $\{ x_n \}_{n \in N}$ is a Cauchy sequence ;
iv. if $q(y, x_n) \leq \alpha_n$ for all $n \in N$, then $\{ x_n \}_{n \in N}$ is a Cauchy sequence ;
v. if $q_1, q_2, q_3 \ldots . q_n$ are Q- functions on X, then $q(x, y) = max \{ q_1(x, y), q_2(x, y), \ldots \ldots , q_n(x, y) \}$ is also a Q- function on X.

Main Result

Throughout this article we denote Ψ the family of non decreasing functions $\Psi : [0,+\infty) \rightarrow [0,+\infty)$ such that $\Sigma_{n=1}^{\infty} \Psi^n(t) < \infty$ for all $t > 0$, where Ψ^n is the n^{th} iterate of Ψ satisfying,

i. $\Psi^{-1}(\{0\}) = \{0\}$,

ii. $\Psi(t) < t$ for all $t > 0$,

iii. $lim_{r \to t^+} \Psi(t) < t$ for all $t > 0$.

Lemma 9:- If $\Psi : [0,\infty] \rightarrow [0,\infty]$ is non decreasing and right continuous, the $\Psi^n(t) \rightarrow 0$ as $n \rightarrow \infty$ for all $t \geq 0$ if and only if $\Psi(t) < t \, for \, all \, t > 0$.

Definition 10:- Let $F: X \times X \rightarrow X$ and $\alpha : X^2 \times X^2 \rightarrow [0,+\infty)$ be two mappings. Then F is said to be $(\alpha) -$ admissible if

$$\alpha\big((x,y),(u,v)\big) \geq 1 \rightarrow \alpha\Big(\big(F(x,y),F(y,x)\big),\big(F(u,v),F(v,u)\big)\Big) \geq 1,$$

for all $x, y, u, v \in X$.

Definition 11:- Let $(X, \leq, d)$ be a partially ordered complete quasi- metric space with a Q- function q on X and $F: X \times X \rightarrow X$ be a mapping. Then a map F is said to be a $(\alpha - \Psi) -$ contractive if there exists two functions $\psi \in \Psi$ and $\alpha : X^2 \times X^2 \rightarrow [0,+\infty)$ such that

$$\alpha\big((x,y),(u,v)\big)q\big(F(x,y),F(u,v)\big) \leq \psi\left(\frac{q(x,u)+q(y,v)}{2}\right)$$

for all $x \geq u \, and \, y \leq v$.

Now we give the main result of this paper, which is as follows.

Theorem 12:- Let $(X, \leq, d)$ be a partially ordered complete $quasi -$ metric space with a $Q -$ function q on X. Suppose that $F : X \times X \rightarrow X$ such that F has the mixed monotone property. Assume that $\psi \in \Psi \, and \, \alpha : X^2 \times X^2 \rightarrow [0,+\infty)$ such that for all $x, y, u, v \in X$ following holds,

$$\alpha\big((x,y),(u,v)\big)q\big(F(x,y),F(u,v)\big) \leq \Psi\left(\frac{q(x,u)+q(y,v)}{2}\right) \qquad 2.1$$

for all $x \leq u \, and \, y \geq v$. Suppose also that

[$(\boldsymbol{a})$]F is $(\alpha) -$ admissible

[$(\boldsymbol{b})$] there exist $x_0, y_0 \in X$ such that

$$\alpha\Big((x_0,y_0),\big(F(x_0,y_0),F(y_0,x_0)\big)\Big) \geq$$

$1 \, and \, \alpha\Big((y_0,x_0),\big(F(y_0,x_0),F(x_0,y_0)\big)\Big) \geq 1$

[$(\boldsymbol{c})$] F is continuous.

If there exists $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0,y_0), \; y_0 \geq F(y_0,x_0)$$

then there exist $x, y \in X$ such that

$$x = F(x,y), \; y = F(y,x) \qquad 2.2$$

that is F has a coupled fixed point.

Proof:- Let $x_0, y_0 \in X$ be such that $\alpha\Big((x_0,y_0),\big(F(x_0,y_0),F(y_0,x_0)\big)\Big) \geq 1$ and $\alpha\Big((y_0,x_0),\big(F(y_0,x_0),F(x_0,y_0)\big)\Big) \geq 1$ and $x_0 \leq F(x_0,y_0) = x\_1$ and $y_0 \geq F(y_0,x_0) = y_1$. Let $x_2, y_2 \in X$ such that $F(x_1,y_1) = x_2$ and $F(y_1,x_1) = y_2$. Continuing this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X as follows,

$$x_{n+1} = F(x_n,y_n) \; and \; y_{n+1} = F(y_n,x_n)$$

for all n ≥ 0. We will show that

$$x_n \leq x_{n+1} \; and \; y_n \geq y_{n+1} \qquad 2.3$$

for all $n \geq 0$. We will use the mathematical induction. Let $n = 0$. Since $x_0 \leq F(x_0, y_0)$, and $y_0 \geq F(y_0, x_0)$ and as $x_1 = F(x_0, y_0)$, and $y_1 = F(y_0, x_0)$. We have $x_0 \leq x_1 \; and \; y_0 \geq y_1$. Thus (2.3) holds for $n = 0$. Now suppose that (2.3) holds for some $n \geq 0$. Then since $x_n \leq x_{n+1}$ and $y_n \geq y_{n+1}$ and by the mixed monotone property of F, we have

$$x_{n+2} = F(x_{n+1}, y_{n+1}) \geq F(x_n, y_{n+1}) \geq F(x_n, y_n) = x_{n+1}$$

and

$$y_{n+2} = F(y_{n+1}, x_{n+1}) \leq F(y_n, x_{n+1}) \leq F(y_n, x_n) = y_{n+1}$$

From above we conclude that

$$x_{n+1} \leq x_{n+2} \; and \; y_{n+1} \geq y_{n+2}$$

Thus by the mathematical induction, we conclude that (2.3) holds for $n \geq 0$. If for some n we have $(x_{n+1}, y_{n+1}) = (x_n, y_n)$, then $F(x_n, y_n) = x_n$ and $F(y_n, x_n) = y_n$ that is, F has a coupled fixed point. Now, we assumed $that \; (x_{n+1}, y_{n+1}) \neq (x_n, y_n)$ for all $n \geq 0$. Since $F \; is \; (\alpha) -$ admissible, we have

$$\alpha\big((x_0, y_0), (x_1, y_1)\big) = \alpha\Big((x_0, y_0), \big(F(x_0, y_0), F(y_0, x_0)\big)\Big) \geq 1$$

$$which\ implies \quad \alpha\Big(\big(F(x_0, y_0), F(y_0, x_0)\big), \big(F(x_1, y_1), F(y_1, x_1)\big)\Big) = \alpha\big((x_1, y_1), (x_2, y_2)\big) \geq 1$$

Thus, by the mathematical induction, we have

$$\alpha\big((x_n, y_n), (x_{n+1}, y_{n+1})\big) \geq 1 \qquad 2.4$$

and similarly,

$$\alpha\big((y_n, x_n), (y_{n+1}, x_{n+1})\big) \geq 1 \qquad 2.5$$

for all $n \in N$. Using (2.1) and (2.4) , we obtain

$$\begin{aligned} q(x_n, x_{n+1}) &= q\big(F(x_{n-1}, y_{n-1}), F(x_n, y_n)\big) \\ &\leq \alpha\left((x_{n-1}, y_{n-1}), (x_n, y_n)\right) q\big(F(x_{n-1}, y_{n-1}), F(x_n, y_n)\big) \\ &\leq \Psi\left(\frac{q(x_{n-1}, x_n) + q(y_{n-1}, y_n)}{2}\right) \end{aligned} \qquad 2.6$$

Similarly we have

$$q(y_n, y_{n+1}) = q\big(F(y_{n-1}, x_{n-1}), F(y_n, x_n)\big)$$

$$\alpha\left((y_{n-1}, x_{n-1}), (y_n, x_n)\right) q\big(F(y_{n-1}, x_{n-1}), F(y_n, x_n)\big) \leq \Psi\left(\frac{q(y_{n-1}, x_n) + q(x_{n-1}, x_n)}{2}\right) \qquad 2.7$$

Adding (2.6)and (2.7), we get

$$\frac{q(x_n, x_{n+1}) + q(y_n, y_{n+1})}{2} \leq \Psi\left(\frac{q(x_{n-1}, x_n) + q(y_{n-1}, y_n)}{2}\right)$$

Repeating the above process, we get

$$\frac{q(x_n, x_{n+1}) + q(y_n, y_{n+1})}{2} \leq \Psi^n\left(\frac{q(x_0, x_1) + q(y_0, y_1)}{2}\right)$$

for all $n \in N$. For $\epsilon > 0 \; there$ exists $n(\epsilon) \in N$ such that

$$\Sigma_{n \geq n(\epsilon)} \Psi^n\left(\frac{q(x_0, x_1) + q(y_0, y_1)}{2}\right) < \frac{\epsilon}{2}$$

Let $m, n \in N$ be such that $m > n > n(\epsilon)$. Then, by using the triangle inequality, we have

$$\begin{aligned} \frac{q(x_n, x_m) + q(y_n, y_m)}{2} &\leq \Sigma_{k=n}^{m-1}\left(\frac{q(x_k, x_{k+1}) + q(y_k, y_{k+1})}{2}\right) \\ &\leq \Sigma_{k=n}^{m-1} \Psi^k\left(\frac{q(x_0, x_1) + q(y_0, y_1)}{2}\right) \\ &\leq \Sigma_{n \geq n(\epsilon)} \Psi^n\left(\frac{q(x_0, x_1) + q(y_0, y_1)}{2}\right) < \frac{\epsilon}{2} \end{aligned}$$

This implies that $q(x_n, x_m) + q(y_n, y_m) < \epsilon$. Since

$$d(x_n, x_m) \leq q(x_n, x_m) + q(y_n, y_m)) < \epsilon$$

and

$$d(y_n, y_m) \leq q(x_n, x_m) + q(y_n, y_m) < \epsilon$$

and hence $\{ x_n \}$ and $\{ y_n \}$ are Cauchy sequences in X. Since (X, d) is complete quasi metric spaces and hence $\{ x_n\}$ and $\{ y_n \}$ are convergent in X. Then there exists $x, y \in X$ such that

$$lim_{n\to\infty} x_n = x \quad lim_{n\to\infty} y_n = y.$$

Since F is continuous and $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$, taking limit $n \to \infty$ we get

$$x = lim_{n\to\infty} x_n = lim_{n\to\infty} F(x_n, y_n) = F(x, y)$$

and

$$y = lim_{n\to\infty} y_n = lim_{n\to\infty} F(y_n, x_n) = F(y, x)$$

that is, $F(x, y) = x$ and $F(y, x) = y$ and hence F has a coupled fixed point.

In the next theorem, we omit the continuity hypothesis of F.

Theorem 13:- Let $(X, \leq, d)$ be a partially ordered complete quasi- metric space with a $Q -$ function q on X. Suppose that $F : X \times X \to X$ such that F has the mixed monotone property. Assume that $\Psi \in \Psi$ and $\alpha : X^2 \times X^2 \to [0, +\infty)$ such that for all $x, y, u, v \in X$ following holds,

$$\alpha\,(\,(x, y), (u, v)\,)q(F(x, y), F(u, v)) \leq \Psi\left(\frac{q(x,u) + q(y,v)}{2}\right) \qquad 2.8$$

for all x ≤ u and y ≥ v. Suppose also that

[$(\boldsymbol{a})$]F is $(\alpha) -$ admissible

[$(\boldsymbol{b})$] there exist $x_0, y_0 \in X$ such that

$$\alpha\Big((x_0, y_0), \big(F(x_0, y_0), F(y_0, x_0)\big)\Big) \geq$$

$$1 \;\; and \;\; \alpha\Big((y_0, x_0), \big(F(y_0, x_0), F(x_0, y_0)\big)\Big) \geq 1$$

[$(\boldsymbol{c})$] if $\{ x_n \}$ and $\{ y_n \}$ are sequences in X such that

$$\alpha\big((x_n, y_n), (x_{n+1}, y_{n+1})\big) \geq 1 \;\; and \;\; \alpha\big((y_n, x_n), (y_{n+1}, x_{n+1})\big) \geq 1$$

for all n and $lim_{n\to\infty} x_n = x \in X$ and $lim_{n\to\infty} y_n = y \in X$, then

$$\alpha\big((x_n, y_n), (x, y)\big) \geq 1 \;\; and \;\; \alpha\big((y_n, x_n), (y, x)\big) \geq 1.$$

If there exists $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0, y_0), \;\; y_0 \geq F(y_0, x_0)$$

then there exist x,y ∈ X such that

$$x = F(x, y), \;\; y = F(y, x) \qquad 2.9$$

that is F has a coupled fixed point.

Proof:- Proceeding along the same line as the above Theorem12, we know $that$ $\{ x_n\}$ and $\{ y_n \}$ are Cauchy sequences in complete quasi metric space X. Then there exists $x, y \in X$ such that

$$lim_{n\to\infty} x_n = x \;\; and \;\; lim_{n\to\infty} y_n = y. \qquad 2.10$$

On the other hand, from (2.4) and hypothesis (c) we obtain

$$\alpha\big((x_n, y_n), (x, y)\big) \geq 1 \qquad 2.11$$

and similarly

$$\alpha\big((y_n, x_n), (y, x)\big) \geq 1 \qquad 2.12$$

for all $n \in N$. Using the triangle inequality, \ref(eq7) and the property of $\Psi(t) < t\, for$ all $t > 0$, we get

$$q(F(x, y), x) \leq q\big(F(x, y), F(x_n, y_n)\big) + q(x_{n+1}, x)$$

$$\leq \alpha\big((x_n, y_n), (x, y)\big) q\big(F(x_n, y_n), F(x, y)\big) + q(x_{n+1}, x)$$
$$\leq \Psi\left(\frac{q(x_n,x) + q(y_n,y)}{2}\right) + q(x_{n+1}, x)$$
$$< \frac{q(x_n,x) + q(y_n,y)}{2} + q(x_{n+1}, x).$$

Similarly, we obtain

$$q(F(y, x), y) \leq q\big(F(y, x), F(y_n, x_n)\big) + q(y_{n+1}, y)$$
$$\leq \alpha\big((y_n, x_n), (y, x)\big) q\big(F(y_n, x_n), F(y, x)\big) + q(y_{n+1}, x)$$
$$\leq \Psi\left(\frac{q(x_n,x) + q(y_n,y)}{2}\right) + q(x_{n+1}, x)$$
$$< \frac{q(y_n,y) + q(x_n,x)}{2} + q(y_{n+1}, y).$$

Taking the limit $n \to \infty$ in the above two inequalities, we get

$$q(F(x, y), x) = 0 \ and \ q(F(y, x), y) = 0.$$

Hence, $F(x, y) = x$ and $F(y, x) = y$. Thus, F has a coupled fixed point.

In the following theorem, we will prove the uniqueness of the coupled fixed point. If $(X, \leq)$ is a partially ordered set, then the product $X \times X$ with the following partial order relation:

$$(x, y) \leq (u, v) \leftrightarrow x \leq u, y \geq v,$$

$for\ all\ (x, y), (u, v) \in X \times X.$

Theorem 14:- In addition to the hypothesis of Theorem 12 suppose that for every $(x, y), (s, t) \in X \times X$, there exists $(u, v) \in X \times X$ such that

$$\alpha((x, y), u, v)) \geq 1 \text{ and } \alpha((s, t), u, v)) \geq 1$$

and also assume that (u, v) is comparable to (x, y) and (s, t). Then F has a unique coupled fixed point.

Proof:- From Theorem 12, the set of coupled fixed point is non empty. Suppose (x, y) and (s, t) are coupled fixed point of the mappings $F: X \times X \to X$, that is $x = F(x, y), y = F(y, x), s = F(s, t)$ and $t = F(t, s)$. By assumption, there exists $(u, v) \in X \times X$ such that (u, v) is comparable to $(x, y)\ and\ (s, t)$. $put\ u = u_0\ and\ v = v_0$ and choose $u_1, v_1 \in X$ such that $u_1 = F(u_0, v_0)\ and\ v_1 = F(v_0, u_0)$. Thus, we can define two sequences $\{ u_n \}$ and $\{ v_n \}$ as

$$u_{n+1} = F(u_n, v_n) \ and \ v_{n+1} = F(v_n, u_n).$$

Since (u, v) is comparable to (x, y), it is easy to show that $x \leq u_1$ and $\geq v_1$. Thus, $x \leq u_n$ and $y \geq v_n$ for all $n \geq 1$. Since for every $(x, y), (s, t) \in X \times X$, there exists $(u, v) \in X \times X$ such that

$$\alpha((x, y), u, v)) \geq 1 \ and \ \alpha((s, t), u, v)) \geq 1. \quad 2.13$$

Since $F\ is\ (\alpha) -$ admissible, so from (2.13), we have

$$\alpha((x, y), u, v)) \geq 1 \to \alpha((F(x, y), F(y, x)), (F(u, v), F(v, u))) \geq 1.$$

Since $u = u_0$ and $v = v_0$, we get

$$\alpha\big((x, y), u_0, v_0\big) \geq 1 \to \alpha((F(x, y), F(y, x)), \big(F(u_0, v_0), F(v_0, u_0)\big) \geq 1.$$

Thus

$$\alpha\big((x, y), (u, v)\big) \geq 1 \to \alpha\big((x, y), (u_1, v_1)\big) \geq 1.$$

Therefore by mathematical induction, we obtain

$$\alpha\big((x, y), (u_n, v_n)\big) \geq 1 \quad 2.14$$

for all $n \in N$ and similarly $\alpha\big((y, x), (v_n, u_n)\big) \geq 1$. From (2.13) and (2.14), we get

$$q(x,u\_(n+1)) = q\big(F(x,y),F(u_n,v_n)\big)$$
$$\leq \alpha\big((x,y),(u_n,v_n)\big)q\big(F(x,y),F(u_n,v_n)\big)$$
$$\leq \Psi\left(\frac{q(x,u_n)+q(y,v_n)}{2}\right).$$

2.15

Similarly, we get

$$q(y,v\_(n+1)) = q\big(F(y,v),F(v_n,u_n)\big)$$
$$\leq \alpha\big((y,x),(v_n,u_n)\big)q\big(F(y,x),F(v_n,x_n)\big)$$
$$\leq \Psi\left(\frac{q(y,v_n)+q(x,u_n)}{2}\right).$$

2.16

Adding (2.15) and (2.16), we get

$$\frac{q(x,u_{n+1})+q(y,v_{n+1})}{2} \leq \Psi\left(\frac{q(x,u_n)+q(y,v_n)}{2}\right)$$

Thus

$$\frac{q(x,u_{n+1})+q(y,v_{n+1})}{2} \leq \Psi^n\left(\frac{q(x,u_1)+q(y,v_1)}{2}\right)$$

2.17

for each $n \geq 1$. Letting $n \to \infty$ in 2.17 and using Lemma 8, we get

$$lim_{n\to\infty}[q(x,u_{n+1}) + q(y,v_{n+1})] = 0$$

This implies

$$lim_{n\to\infty}q(x,u_{n+1}) = 0 \;\; lim_{n\to\infty}q(y,v_{n+1}) = 0.$$

2.18

Similarly we can show that

$$lim_{n\to\infty}q(s,u_{n+1}) = 0 \;\; lim_{n\to\infty} q(t,v_{n+1}) = 0.$$

2.19

From 2.18 and 2.19, we conclude that $x = s$ and $y = t$. Hence, F has a unique coupled fixed point.

Example 15:- Let $X = [0,1]$, with the usual partial ordered $\leq$. Defined $d: X \times X \to R^+$ by

$$d(x,y) = \begin{cases} y-x \;\; if \;\; x=y \\ 2(x-y) \;\; otherwise \end{cases}$$

and $q: X \times X \to R\^+$ by

$$q(x,y) = |\; x - y \;|, \; \forall\, x,y \in X.$$

2.20

Then d is a quasi metric and q is a $Q-$ function on X. Thus, $(X,d,\leq)$ is a partially ordered complete quasi metric space with Q- function q on X.

Consider a mapping $\alpha : X^2 \times X^2 \to [0,+\infty)$ be such that

$$\alpha\big((x,y),(u,v)\big) = \begin{cases} 1 \;\; if \; x \geq y, u \geq v \\ \;\; 0 \;\; otherwise \end{cases}$$

Let $\Psi\,(t) = \frac{t}{2}$, for $t > 0$. Defined $F: X \times X \to X$ by $F(x,y) = \frac{1}{4}xy$ for all $x,y \in X$.

Since $|\; xy - uv \;| \leq |\; x - u \;| + |\; y - v \;|$ $holds$ for all $x,y,u,v \in X$. Therefore, we have

$$q\big(F(x,y),F(u,v)\big) = |\; \frac{xy}{4} - \frac{uv}{4} \;|$$
$$\leq \frac{1}{4}(|\; x-u \;| + |\; y-v \;|)$$
$$= \frac{1}{4}\left(q(x,u) + q(y,v)\right)$$

It follows that, $\alpha\,(\,(x,y),(u,v)\,)q(F(x,y),F(u,v)) \leq \frac{1}{4}\,(q(x,u) + q(y,v))$

Thus \ref(eq1) holds for $\Psi(t) = \frac{t}{2}$ for all $t > 0$ and we also see that all the hypothesis of Theorem 12are fulfilled. Then there exists a coupled fixed point of F. In this case (0,0) is coupled fixed point of F.

Example 16:- Let $X = [0,1]$, with the usual partial ordered $\leq$. Defined $d: X \times X \to R^{+}$ by

$$d(x,y) = \begin{cases} y - x \ if \ x = y \\ 2(x-y) \ otherwise \end{cases}$$

and $q: X \times X \to R^{+}$ by

$$q(x,y) = |\, x - y \,|, \ \forall\, x,y \in X$$

2.21

Then d is a quasi metric and q is a $Q-$ function on X. Thus, $(X,q,\leq)$ is a partially ordered complete quasi metric space with Q- function q on X.

Consider a mapping $\alpha : X^2 \times X^2 \to [0,+\infty)$ be such that

$$\alpha\big((x,y),(u,v)\big) = \begin{cases} 1 \ \ if\ x \geq y, u \geq v \\ \ \ 0 \ \ otherwise \end{cases}$$

Let $(t) = 2\,t, for\ t > 0$. Defined $F: X \times X \to X$ by $F(x,y) = sin\,x + sin\,y$ for $all\ x,y \in X$. Since $|\, sin\,x - sin\,y \,| \leq |\, x - y \,|$ holds for all $x,y \in X$. Therefore, we have

$$\begin{aligned} q\big(F(x,y),F(u,v)\big) &= |\, sin\,x + sin\,y - sin\,u - sin\,v \,| \\ &\leq |\, sin\,x - sin\,u \,| + |\, sin\,y - sin\,v \,| \\ &\leq |\, x - u \,| + |\, y - v \,| \\ &\leq \Psi\left(\frac{(q(x,u) + q(y,v))}{2}\right). \end{aligned}$$

Then there exists a coupled fixed point of F. In this case (0,0) is coupled fixed point of F.

Corollary 17:- Let $(X,\leq,d)$ be a partially ordered complete quasi- metric space with a $Q-$ function q on X. Suppose that $F : X \times X \to X$ such that F is continuous and has the mixed monotone property. Assume that $\Psi \in \Psi$ and such that for all $x,y,u,v \in X$ following holds,

$$q(F(x,y),F(u,v)) \leq \Psi\left(\frac{q(x,u) + q(y,v)}{2}\right)$$

2.22

for all $x \leq u$ and $y \geq v$.

If there exists $x\_0, y\_0 \in X$ such that

$$x_0 \leq F(x_0,y_0), \ y_0 \geq F(y_0,x_0)$$

then there exist $x,y \in X$ such that

$$x = F(x,y), \ y = F(y,x)$$

2.23

that is F has a coupled fixed point.

Proof:- It is easily to see that if we take $\alpha\,(\,(x,y),(u,v)\,) = 1$ in Theorem 12 then we get Corollary 17.

Corollary 18:- Let $(X,\leq,d)$ be a partially ordered complete quasi- metric space with a Q- function q on X. Suppose that $F : X \times X \to X$ such that F is continuous and has the mixed monotone property. Assume that $\Psi \in \Psi$ and such that for all $x,y,u,v \in X$ following holds,

$$q(F(x,y),F(u,v)) \leq \frac{k}{2}[q(x,u) + q(y,v)]$$

2.24

for $k \in [0,1)$ and for all $x \leq u$ and $y \geq v$.

If there exists $x\_0, y\_0 \in X$ such that

$$x_0 \leq F(x_0, y_0), \; y_0 \geq F(y_0, x_0)$$

then there exist $x, y \in X$ such that

$$x = F(x, y), \; y = F(y, x)$$

2.25

that is F has a coupled fixed point.

Proof:- It is easily to see that if we take $\Psi(t) = kt$ in Corollary 17 then we get Corollary 18.

Corollary 19:- *Let* $(X, \leq, d)$ be a partially ordered complete quasi metric space with a Q-function q on X. Assume that the function $\Psi : [0, +\infty) \to [0, +\infty)$ is such *that* $\Psi(t) < t$ for each $t > 0$. Further suppose that $F: X \times X \to X$ is such that F has the mixed monotone property and

$$q(F(x, y), F(u, v)) \leq \Psi\left(\frac{q(x,u) + q(y,v)}{2}\right)$$

2.29

for all $x, y, u, v \in X$ for which $x \leq u$ and $y \leq v$. Suppose that F satisfies following,

[(a)]F is continuous or

[(b)] X has the following property:

[(i)] if a non decreasing sequence $\{x\_n\} \to x$ then $x\_n \leq x$ for all n,

[(ii)] if a non increasing sequence $\{y\_n\} \to y$ then $y\_n \geq y$ for all n.

If there exists $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0, y_0), \; y_0 \geq F(y_0, x_0)$$

2.30

then there exist $x, y \in X$ such that

$$x = F(x, y), \; y = F(y, x)$$

2.31

that is F has a coupled fixed point.

Proof:- It is easily to see that if we take $\alpha\big((x, y), (u, v)\big) = 1$ and from the property in Theorem 12 then we get Corollary 19.

Corollary20 :- Let $(X, \leq, d)$ be a partially ordered complete quasi metric space with a Q-function q on X. Assume that the function $\Psi : [0, +\infty) \to [0, +\infty)$ is such that $\Psi(t) < t$ for each $t > 0$. Further suppose that $F: X \times X \to X$ is such that F has the mixed monotone property and

$$q\big(F(x, y), F(u, v)\big) \leq \frac{k}{2}[q(x, u) + q(y, v)]$$

2.32

for all $k \in [0,1), x, y, u, v \in X$ for which $x \leq u$ and $y \leq v$. Suppose that F satisfies following,

[(a)]F is continuous or

[(b)] X has the following property:

[(i)] if a non decreasing sequence $\{x_n\} \to x$ then $x_n \leq x$ for all n,

[(ii)] if a non increasing sequence $\{y_n\} \to y$ then $y_n \geq y$ for all n.

If there exists $x_0, y_{}0 \in X$ such that

$$x_0 \leq F(x_0, y_0), \; y_0 \geq F(y_0, x_0)$$

then there exist $x, y \in X$ such that

$$x = F(x, y), \; y = F(y, x)$$

2.33

that is F has a coupled fixed point.

Proof:- It is easily to see that if we take $\Psi(t) = kt$ in Theorem 12 then we get Corollary 20.

Now our next result show that $(\alpha) -$ admissible function is work like as a control function, but converges may not be true in general. We also give an example in support of this fact.

Theorem 21:- Let $(X, \leq, d)$ be a partially ordered complete quasi- metric space with a Q- function q on X. Suppose that $F : X \times X \to X$ such that F has the mixed monotone property. Assume that $\alpha: X^2 \times X^2 \to [0, +\infty)$ such that for all $x, y, u, v \in X$ following holds,

$$\alpha((x,y),(u,v))q(F(x,y),F(u,v)) \leq \frac{k}{2}[q(x,u) + q(y,v)] \quad 2.34$$

for $k \in [0,1)$ and for all $x \leq u$ and $y \geq v$. Suppose also that

[(a)]F is $(\alpha) -$ admissible

[(b)] there exist $x_0, y_0 \in X$ such that

$$\alpha\left((x_0,y_0),\left(F(x_0,y_0),F(y_0,x_0)\right)\right) \geq 1$$

and

$$\alpha\left((y_0,x_0),\left(F(y_0,x_0),F(x_0,y_0)\right)\right) \geq 1$$

[(c)] F is continuous.

If there exists $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0,y_0), \; y_0 \geq F(y_0,x_0)$$

then there exist $x, y \in X$ such that

$$x = F(x,y), \; y = F(y,x) \quad 2.35$$

that is F has a coupled fixed point.

Proof:- If we take $\Psi(t) = kt$ in Theorem 12 then the remaining prove of the above Theorem 21 is similar to the prove of Theorem 12.

Example 22:- Let $X = [0, \infty)$, with the usual partial ordered $\leq$. Defined $d: X \times X \to R^+$ by

$$d(x,y) = \begin{cases} y - x & if \; x = y \\ 2(x-y) & otherwise \end{cases}$$

and $q: X \times X \to R^+$ by

$$q(x,y) = | x - y |, \; \forall x, y \in X.$$

Then d is a quasi metric and q is a Q- function on X. Thus, $(X, d, \leq)$ is a partially ordered complete quasi metric space with Q- function q on X.

Consider a mapping $\alpha : X^2 \times X^2 \to [0, +\infty)$ be such that

$$\alpha((x,y),(u,v)) = \begin{cases} 1 & if \; x \geq y, u \geq v \\ 0 & otherwise \end{cases}$$

Defined $F: X \times X \to X$ by

$$F(x,y) = \begin{cases} \frac{x-y}{2} & if \; x \leq y \\ 0 & otherwise \end{cases}$$

Then there is no any $k \in [0,1)$ for which satisfying all conditions of Theorem \ref(thm4).

If we take $\alpha : X^2 \times X^2 \to [0, +\infty)$ as follows,

$$\alpha((x,y),(u,v)) = \begin{cases} 2 & if \; x \geq y, u \geq v \\ 0 & otherwise \end{cases}$$

Then there is $k = \frac{1}{2} \in [0,1)$ such that all conditions of Theorem 21 are satisfies and $(0,0)$ is a coupled fixed point of F.

Acknowledgements:
The authors would like to express their sincere thanks to the editor and the anonymous referees for their valuable comments and useful suggestions in improving the article.

References

1. S. Al-Homidan, Q. H. Ansari, and J.-C. Yao, “Some generalizations of Ekeland-type variational principle with applications to equilibrium problems and fixed point theory,” Nonlinear Analysis: Theory, Methods and Applications, vol. 69, no. 1, pp. 126--139, 2008.
2. Q. H. Ansari, “Vectorial form of Ekeland-type variational principle with applications to vector quilibrium problems and fixed point theory,” Journal of Mathematical Analysis and Applications, vol. 334, no. 1, pp. 561--575, 2007.
3. Q. H. Ansari, I. V. Konnov, and J. C. Yao, “On generalized vector equilibrium problems,” Nonlinear Analysis: Theory, Methods and Applications, vol. 47, no. 1, pp. 543--554, 2001.
4. Q. H. Ansari, A. H. Siddiqi, and S. Y. Wu, “Existence and duality of generalized vector equilibrium problems,” Journal of Mathematical Analysis and Applications, vol. 259, no. 1, pp. 115--126, 2001.
5. Q. H. Ansari and J.-C. Yao, “An existence result for the generalized vector equilibrium problem,” Applied Mathematics Letters, vol. 12, no. 8, pp. 53 --56, 1999.
6. Q. H. Ansari and J.-C. Yao, “A fixed point theorem and its applications to a system of variational inequalities,” Bulletin of the Australian Mathematical Society, vol. 59, no. 3, pp. 433--442, 1999.
7. A. D. Arvanitakis, A proof of the generalized Banach contraction conjecture. Proc. Am. Math. Soc. 131(12, 3647--3656 (2003).
8. S. Banach, Sur les op'(erations dans les ensembles abstraits et leurs applications aux '(equations i'(egrales, Fund. Math. 3 (1922 133-181.).
9. T.G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal. TMA 65, 1379--1393 (2006).
10. D. W. Boyd, J. S. W. Wong, On nonlinear contractions. Proc. Am. Math. Soc. 20, 458--464 (1969).
11. Dz. Burgic, S. Kalabusic, M.R.S. Kulenovic, Global attractivity results for mixed monotone mappings in partially ordered complete metric spaces, Fixed Point Theory Appl. (2009 Article ID 762478.)
12. Lj. B. Ciric, N. Cakid, M. Rajovic, J.S, Ume, Monotone generalized nonlinear contractions in partially ordered metric spaces, Fixed Point Theory Appl./(2008 Article ID 131294.
13. J. Harjani, K. Sadarangani, Generalized contractions in partially ordered metric spaces and applications to ordinary differntial equations, Nonlinear Anal. 72, 1188--1197 (2010.).
14. D. Guo, V. Lakshmikantham, Coupled fixed points of nonlinear operators with applications. Nonlinear Anal., Theory Methods Appl. 11 (1987)623--632.
15. O. Kada, T. Suzuki, and W. Takahashi, “Nonconvex minimization theorems and fixed point theorems in complete metric spaces,” Mathematica Japonica, vol. 44, no. 2, pp. 381–391, 1996.
16. L.-J. Lin and W.-S. Du, “Ekeland’s variational principle, minimax theorems and existence of nonconvex equilibria in complete metric spaces,” Journal of Mathematical Analysis and Applications, vol. 323, no. 1, pp. 360-370, 2006.
17. S.G. Matthews, Partial metric topology, Ann. New York Acad. Sci. vol. 728 183-197 , 1994.

18. J.J. Nieto, R. R. Lopez, Contractive mappin theorems in partially ordered sets and applications to ordinary differential equations, Order, 22, 223--239 (2005.)
19. J.J. Nieto, R. R. Lopez, Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations, Acta Math.Sin. (Engl. Set. 23, 2205 -- 2212 (2007.)
20. S. Romaguera, M. Schellekens, Partial metric monoids and semivaluation spaces, Topol. Appl. vol. 153, 948-962, 2005.)
21. S. Romaguera, M.P. Schellekens, O. Valero, Complexity spaces as quantitative domains of computation, Topology Appl. vol. 158, 853-860, 2011.)
22. S. Romaguera, P. Tirado, O. Valero, Complete partial metric spaces have partially metrizable computational models, Int. J. Comput. Math. 89, 284-290, 2012.)
23. S. Romaguera, O. Valero, A quantitative computational model for complete partial metric spaces via formal balls, Math. Struc. Comp. Sci. vol. 19, 541-563, 2009.)
24. S. Romaguera, O. Valero, Domain theoretic characterizations of quasimetric completeness in terms of formal balls, Math. Struc. Comp. Sci. vol. 20, 453-472, 2010).
25. W. Sintunavarat, Y.J. Cho, P. Kumam, Coupled fixed point theorems for weak contraction mapping under F-invariant set. Abstr. Appl. Anal. 2012, 15 (Article ID 324874 (2012.)
26. W. Sintunavarat, P. Kumam,: Gregus type fixed points for a tangential multi-valued mappings satisfying contractive conditions of integral type. J. Inequal. Appl. 2011, 3 (2011.)
27. W. Sintunavarat, Y.J. Cho, Kumam, P: Common fixed point theorems for c-distance in ordered cone metric spaces. Comput. Math. Appl. 62, 1969--1978 (2011.)
28. W. Sintunavarat, P. Kumam,: Common fixed point theorems for hybrid generalized multivalued contraction mappings. Appl. Math. Lett. 25, 52--57 (2012).
29. W. Sintunavarat, P. Kumam,: Common fixed point theorems for generalized operator classes and invariant approximations. J. Inequal. Appl. 2011, 67 (2011).
30. W. Sintunavarat, P. Kumam,: Generalized common fixed point theorems in complex valued metric spaces and applications. J. Inequal. Appl. 2012, 84 (2012).
31. T. Suzuki,: A generalized Banach contraction principle that characterizes metric completeness. Proc. Am. Math. Soc. 136(5, 1861--1869 (2008).