International Journal for Multidisciplinary Research (IJFMR)
E-ISSN: 2582-2160 ● Website: www.ijfmr.com ● Email: editor@ijfmr.com

# Geometry of Statistical Sequential Warped Product Manifolds

**Bernard Kankonde Ilunga**

Département de Mathématiques, Université Pédagogique de Kananga, B.P. 282-Kananga, République Démocratique du Congo.

**Abstract**
The main purpose of the present study is to generalize the dualistic structures, specially the statistical structures on warped product manifolds to statistical structures on sequential warped product manifolds.On the one hand we proved that the statistical structure on sequential warped product manifold induces statistical on the factors and conversely.On the other hand we proved that if the statistical warped product is dually at space, then the base manifold is also dually at and ber is a space form, i.e. of constant sectional curvature.
MSC (2010): 53B05, 53B25, 53A40. Key words and phrases: dual connections, warped product of manifolds, sequential warped product of manifolds, statistical structures,Riemannian manifolds.
Introduction
In [1], S. Amari studied in 1985 the notion of statistical manifolds.The abstract generalization of statistical models are considered as the statistical manifolds.The geometry of these manifolds lies the junction of several branches of geometry: information geometry, a ne differential geometry, and Hessian geometry.... Statistical structure can be considered as generalization of Riemannian structure with a pair of Riemannian metric and its Levi-Civita connection(See [2]). The triple $(M,\nabla,g)$ is called a statistical manifold if

1) $\nabla$ is of torsion free and
2) $(\nabla g)$ is symmetric.

The pair $(\nabla,g)$ is called statistical structure on Riemannian manifold M. The affine connection $\nabla^*$ of $M$ is called dual connection of $\nabla$ with respect to $g$ if

$$X.g(Y,Z) = g(\nabla_X Y) + g(Y, \nabla^*_X Z)$$

for all $X,Y,Z \in \chi(M)$.

The triple $(g, \nabla, \nabla^*)$ satsfying the previous relation is called dualistic structure on $M$. (*1*) and (*2*) are equivalent to $\nabla$ and $\nabla^*$ are torsion-free. Trivially, we have $(\nabla^*)^* = \nabla$.

Dualistic structures play an important role in the investigation of the natural differential geometric structure posseded by families of probability distributions.

$(M, g, \nabla, \nabla^*)$ is said to be a dually flat space if both dual connections $\nabla$ and $\nabla^*$ are torsion free and flat; that is the curvature tensors with respect to $\nabla$ and $\nabla^*$ respectively vanishe identically.

In [9] the author proved that the projection of dualistic structure defined on a warped product spaces induces structures on the base and the fiber. Recently in [4,6] it is proved that the projection of dualistic (statistical) structure on a doubly warped product spaces induces the same structure on the base and on the fiber, and conversely dualistic (statistical) structure on the base and the fiber induces the structure on the doubly warped product space.

The concept of warped product was first introduced by O'Neill and Bishop to construct examples of Riemannnian manifolds with negative curvature. Let $(B, g_B)$ and $(F, g_F)$ be two Riemannian manifolds and $f$ be smooth positive function on $B$.

Consider the product manifold $B \times F$ with its canonical projections $\pi : B\times F \to B$ and $\sigma : B\times F \to F$. The warped product $M = B \times_f F$ is the manifold with Riemannian structure such that $g = g_B \oplus f^2 g_F$.

Here $B$ is called the base space, $F$ the fiber, and $f$ is the warping function of $M$ [7].

Now, we can generalize warped products to doubly and multipy warped product (see [10]). The doubly warped product manifold $B \times_{f_1 f_2} F$ is defined as the product manifold $B \times F$ endowed with Riemannian metric which is denoted by $g_{f_1 f_2}$, given by: $g_{f_1 f_2} = (f_2 \circ \sigma)^2 \pi^* g_1 + (f_1 \circ \pi)^2 \sigma^* g_2$. In this case $f_1 = 1$ or $f_2 = 1$ we obtain a warped product or a direct product. A multiply warped product manifolds is the product $M = B \times_{b_1} F_1 \times_{b_2} F_2 \ldots \times_{b_m} F_m$ with the metric $g$ defined by $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \oplus \ldots \oplus b_m^2 g_{F_m}$ where $b_i : B \to (0, \infty)$ and $(F_i, g_{F_i})$ a Riemannian manifolds for $1 \leq i \leq m$.
In [8] S.Shenawy introduced another generalization of warped product manifolds called sequential warped product manifolds and the geometry of such warped products have been studied in details by De, Shenawy and Unal in [3]. The sequential warped products are a suitable structure for expressing generalized Robertson-Walker space-time and standard static-space-time.
In the present paper, we are interested in statistical structure on sequential warped product manifolds. In Section 2, we provide a brief of the notion of statistical manifolds and sequential warped product maniolds. In Section 3 after recalling some properties of statistical structure, we study statistical structure on sequential warped product manifolds.

## 1 Preliminaries

### 1.1 Statistical manifolds

Let $M$ be a smooth manifold of dimension $n$ with $n \geq 2$, $\nabla$ an a ne connection on $M$ and $g$ a Riemanniann metric $M$.
1) $(M, \nabla, g)$ is called a statistical manifold if:
$\nabla$ is torsion-free, and
$(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z)$.
2) $\nabla^*$ is the dual (conjugate) connection of $\nabla$ with respect to $g$ if:

$$X.g(Y, Z) = g(\nabla_X Y) + g(Y, \nabla^*_X Z).$$

Moreover, the triple $(M, \nabla^*, g)$ is said to be the dual statistical manifold of $(M, \nabla, g)$ and the triple $(\nabla, \nabla^*, g)$ is called the dualistic structure on $M$.
Due to fact that $(\nabla^*)^* = \nabla$ the affine connections $\nabla$ and $\nabla^*$ are dual connections. It is easy to see that the dual connections $\nabla$ and $\nabla^*$ are related by $\nabla + \nabla^* = 2\nabla^\circ$, where $\nabla^\circ$ is Levi-Civita connection of the metric $g$.
**Proposition 1.1**. *A triple $(M, \nabla, g)$ is a statistical manifold if and only if $(M, \nabla^*, g)$ is a statistical manifold.*
Proof. See [6] □
Hence, the geometry of statistical manifolds simply reduces to the semi Riemannian geometry when $\nabla$ and $\nabla^*$ coincide. In this case, the pair $(\nabla^0, g)$ is said to be a Riemannian statistical structure or trivial statistical structure.
Denote $R$ and $R^*$ the curvature tensor fields of $\nabla$ and $\nabla^*$ respectively. A statistical structure is said to be of constant curvature $c \in$ R if:

$$R(X, Y)Z = c\,(g(Y, Z)X - g(X, Z))\,Y.$$

The curvature tensor fields $R$ and $R^*$ satisfy

$$g(R^*(X, Y)Z, W) = -g(Z, R(X, Y)W)$$

A statistical structure $(\nabla, g)$ of constant curvature 0 is called Hessian structure.
**Proposition 1.2**. *A statistical manifold $(M, \nabla, g)$ is of constant curvature $c$ if and only if $(M, \nabla^*, g)$ is of constant curvature $c$.*

### 1.2 Statistical manifolds on warped product manifolds

Let $(M_1, g_1)$ and $(M_2, g_2)$ be two Riemannian manifolds of dimension $m$ and $n$ respectively and $f$ a smooth positive function on $M_1$.

The warped product manifolds of $(M_1, g_1)$ and $(M_2, g_2)$ with warping function $f$ is the $(m+n)$-dimensional manifold $M_1 \times M_2$ endowed with the metric $g$ given by: $g = \pi^* g_1 \oplus (f \circ \pi)^2 \sigma^* g_2$ where $\pi : M_1 \times M_2 \to M_1$ and
$\sigma : M_1 \times M_2 \to M_2$ their canonical projections.
The tangent space $T_{(p,q)}(M_1 \times M_2)$ is isomorphic to the direct sum $T_pM_1 \oplus TqM2$.
Let $L_hM_1$ (resp.$L_vM_2$) be the set of all vector fields on $M_1 \times M_2$, each of which is the horizontal lift (resp. the vertical lift) of a vector fields on $M_1$ (resp. on $M_2$.)
We have $T(M_1 \times M_2) = L_hM_1 \oplus L_vM_2$ and thus a vector field $A$ on $M_1 \times M_2$ can be written as $A = X + U$ with $X \in L_hM_1$ and $U \in L_vM_2$.
Obviously, $\pi_*(L_hM_1) = TM_1$ and $\sigma_*(L_vM_2) = TM_2$. For any vector field $X \in L_hM_1$, denote $\pi_*(X)$ by $\bar{X}$, and for any vector field $U \in L_vM_2$ we denote $\sigma_*(U)$ by
$\tilde{U}$.
Let $(\nabla^1, \nabla^{*1})$, $(\nabla^2, \nabla^{*2})$, and $(\nabla, D^*)$ be dualistic structures on $M_1$, $M_2$, and $M_1 \times M_2$ respectively. For any $X, Y \in L_hM_1$ and $U, V \in L_vM_2$

$$\pi_*(\nabla_X Y) = \nabla^1_{\bar{X}} \bar{Y} \quad \text{and} \quad \pi_*(\nabla^*_X Y) = \nabla^{*1}_{\bar{X}} \bar{Y}. \tag{1.1}$$

$$\sigma_*(\nabla_U V) = \nabla^2_{\tilde{U}} \tilde{V} \quad \text{and} \quad \sigma_*(\nabla^*_U V) = \nabla^{*2}_{\tilde{X}} \tilde{Y} \tag{1.2}$$

**Lemma 1.1.** *Let $\bar{X}, \bar{Y}, \bar{Z} \in \Gamma(TM_1)$ and $X, Y, Z \in L_hM_1$ be their corresponding horizontal lifts respectively. Let $\tilde{U}, \tilde{V}, \tilde{W} \in \Gamma(TM_2)$ and $U, V, W \in L_vM_2$ be their corresponding vertical lifts respectively. Then:*

$$X.\bar{g}(\bar{Y}, \bar{Z}) \circ \pi = X.g(Y,Z) \tag{1.3}$$

$$U.\tilde{g}(\tilde{V}, \tilde{W}) \circ \sigma = U.g(V,W) \tag{1.4}$$

See [10].
**Proposition 1.3**. *Let $(g, \nabla, \nabla^*)$ be a dualistic structure on a warped product $M_1 \times_f M_2$. Then the projections induce dualistic structure on the base and the fiber manifolds.*
**Proposition 1.4.** *A triple $(M_1, \nabla^1, g_1)$ and $(M_2, \nabla^2, g_2)$ are statistical manifolds if and only if $(M_1 \times_f M_2, \nabla, g)$ is a statistical manifold.*
Proof. In [4] if $f_2 = 1$, doubly warped product manifolds becomes warped product manifolds with $f_1$ a warping function.The rest of proof follows as in
[9]. □
**Proposition 1.5.** *Let $(M_1, \nabla^1, g_1)$ a warped product manifold and R Riemannian curvature tensor on M. If X, Y, Z are tangent to leaves. Then $R_{XY}Z$ is horizontal.*
Proof. We know that $R_{XY}Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$. by lemma B in [9] the proof is direct because [X,Y] is horizontal, in the same way $[\nabla_X, \nabla_Y]Z$ and $\nabla_{[X,Y]}Z$ are horizontal. □

## 2 Statistical structures on sequential warped product manifolds

### 2.1 Sequential warped products

Let $M_i$ be three peudo-riemannian manifolds with metrics $g_i$ for $i = 1, 2, 3$. Let $f : M_1 \to (0, \infty)$ and $h : M_1 \times M_2 \to (0, \infty)$ be two smooth positive functions on $M$ and $M_1 \times M_2$ repectively.
**Definition 2.1**. The sequential warped product manifolds, denoted by $(M_1 \times_f M_2) \times_h M_3$ is the triple product manifolds $(M_1 \times M_2) \times M_3$ furnished with the metric tensor $g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$ where $f$ and $h$ are the warping functions.
If $(M_i, g_i)$ are all Riemannian manifolds for any $i = 1,2,3$ then the sequential warped product manifolds $(M_1 \times_f M_2) \times_h M_3$ is also a Riemannian manifold. The warped product of the form $M_1 \times_{f1} (M_2 \times_{f2} M_3)$ equipped with the metric $g = g_1 \oplus f_1^2(g_2 \oplus f_2^2 g_3)$ is called the iterated warped product of
$M_1$, $M_2$, and $M_3$. As a metric space, the iterated warped product manifolds is equal to sequential warped product $(M_1 \times_f M_2) \times_h M_3$ where $f = f_1$ and $h = f_1 f_2$.

Similary, a sequential warped product $(M_1 \times_f M_2) \times_h M_3$ with separable function $h : M_1 \times M_2 \to (0,\infty)$ is equal as a space to the iterated warped product manifolds. If the warping function $h$ of the sequential warped product$(M_1 \times_f M_2) \times_h M_3$ is defined only on $M_1$, then we have a multiply warped product manifold $M_1 \times_f M_2 \times_h M_3$ with two fibers $M_2$ and $M_3$.
The multiply warped product manifolds of the form $(M_1 \times_{f1} M_2) \times_{f2} M_3$ equipped with the metric $g$ given by: $g = (g_1 \oplus f_1^2 g_2) \oplus f_2^2 g_3$ where both $f_1$ and $f_2$ are positive functions on $M_1$.
**Proposition 2.1.** *Let* $\bar{M} = (M_1 \times_f M_2) \times_h M_3$. *be a sequential warped product manifolds with metric:*
$\bar{g} = (g_1 + f^2 g_2) + h^2 g_3$
*and also let* $X_i, Y_i \in \Gamma(TM_i)$. *Then:*
(1) $\bar{\nabla}_{X1}Y1 = \nabla^1_{X1}Y1$;
(2) $\bar{\nabla}_{X1}X2 = \bar{\nabla}_{X2}X1 = X1(\ln f)$;
(3) $\bar{\nabla}_{X2}Y2 = \nabla^2_{X2}Y2 - fg_2(X2,Y2)grad^1 f$;
(4) $\bar{\nabla}_{X3}Y1 = \bar{\nabla}_{X1}X3 = X1(\ln h)X3$;
(5) $\bar{\nabla}_{X2}X3 = \bar{\nabla}_{X3}X2 = X2(\ln h)X3$;
(6) $\bar{\nabla}_{X_3}Y_3 = \nabla^3_{X_3}Y_3 - hg_3(X_3,Y_3)grad h$.
(7) $\bar{\nabla}^*_{X_1}Y_1 = \nabla^{*1}_{X_1}Y_1$;
(8) $\bar{\nabla}^*_{X1}X2 = \bar{\nabla}^*_{X2}X1 = X1(\ln f)$;
(9) $\bar{\nabla}^*_{X_2}Y_2 = \nabla^{*2}_{X_2}Y_2 - fg_2(X_2,Y_2)grad^1 f$;
(10) $\bar{\nabla}^*_{X3}Y1 = \bar{\nabla}^*_{X1}X3 = X1(\ln h)X3$;
11) $\bar{\nabla}^*_{X_2}X_3 = \bar{\nabla}^*_{X_3}X_2 = X_2(\ln h)X_3$; 12) $\bar{\nabla}^*_{X_3}Y_3 = \nabla^{*3}_{X_3}Y_3 - hg_3(X_3,Y_3)grad h$.
Proof. See in [8] □

**Proposition 2.2.** Let $(\bar{g}, \nabla, \nabla^*)$ be a dualistic structure on $\bar{M}$. Then there exists an affine connections $\nabla^i, \nabla^{i*}$ *on* $M_i$, such that $(g_i, \nabla^i, \nabla^{i*})$ is a dualistic structure on $M_i$ for $i = 1,2,3$.
Proof. Let $\bar{M} = (M_1 \times_f M_2) \times_h M_3$, with metric $\bar{g}$ and Riemannian connection $\nabla$. For all $X_i, Y_i, Z_i \in \Gamma(TM)$. we assume that:

$$X_i(\bar{g}(Y_i,Z_i)) = \bar{g}(\nabla_{X_i}Y_i,Z_i) + \bar{g}(Y_i,\nabla^*_{X_i}Z_i).$$

And

$$\bar{g}(Y,Z_i) = (f_i)^2 g_i(Y_i,Z_i),$$

for $f_1 = 1, f_2 = f$ and $f_3 = h$, then:
$X.\bar{g}(Y,Z_i) = (f_i)^2 X_i g_i(Y_i,Z_i)$ for $X \in \bar{M}$, this equation is equivalent to

$$(f_i)^2 X_i.g_i(Y_i,Z_i) = (f_i)^2 \{g(\nabla_{X_i}Y_i,Z_i) + g(Y_i,\nabla^*_{X_i}Z_i)\}.$$

Thus these a ne connections $\nabla^i, \nabla^{*i}$ are conjugate with respect to $g_i$. □
Proposition 2.3. Let $(g_i, \nabla^i, \nabla^{i*})$ is a dualistic structure on $M_i$ for $i =$ 1,2,3. Then there exists a dualistic structure on $\bar{M} = (M_1 \times_f M_2) \times_h M_3$. with respect to $\bar{g}$.
Proof. Two connections are conjugate with respect to Riemannian metric $g$ if the cubic tensor $C(X,Y,Z) := \nabla g(X,Y,Z) = 0$ i.e for $X_i, Y_i, Z_i \in \chi(M_i)$ with $i = 1,2,3$ and $f_1 = 1, f_2 = f, f_3 = h$. We have :

$$\begin{aligned} X_i(\overline{g}(Y_i,Z_i)) &= X_i((f_i)^2 g_i(Y_i,Z_i)) \\ &= (f_i)^2 X_i g_i(Y_i,Z_i) \\ &= (f_i)^2 \{g_i(\nabla_{X_i}Y_i,Z_i) + g_i(Y_i,\nabla^*{}_{X_i}Z_i)\} \end{aligned}$$

Because $(g_i, \nabla^i, \nabla^{i*})$ is a dualist structure, and on the other hand we have:

$$\bar{g}(\nabla_{X_j}Y_i,Z_i) + \bar{g}(Y_i,\nabla^*_{X_j}Z_i) = (f_i)^2 \{g_i(\nabla_{X_i}Y_i,Z_i) + g_i(Y_i,\nabla^*_{X_i}Z_i)\}$$

, it is clear that $C(X_1,Y_1,Z_1) = 0$. if $j \neq i$ we get:

$$X_j(\bar{g}(Y_i,Z_i)) = 2(f_i)X_j(f_i)g_1(Y_i,Z_1)$$

$g(\nabla_{X_j}Y_i,Z_i) = f_i X_j(f_i) g_i(Y_i,Z_i)$ and $g(Y_i,\nabla^*_{X_j}Z_i) = f_i X_j(f_i) g_i(Y_i,Z_i)$. Then

$$C(X_j,Y_i,Z_i) = C(X_i,Y_j,Z_i) = C(X_i,Y_i,Z_j) = 0,$$

the connections $\nabla$ and $\nabla^*$ are dualistic with respect to $g$. □

**Proposition 2.4.** *Let $(M_i, \vec{\nabla}, g_i)$ are statistical manifolds if and only if $(M, \nabla, g)$ is a statistical manifold.*

Proof. First of all we know that the connection $\nabla$ on the sequential product manifolds is torsionless on $M_i$ because $\nabla^1$, $\nabla^2$,and $\nabla^3$ are torsionless.

$$g(\nabla_X Y) = g(\nabla_{X1}Y,Z) + g(\nabla_{X2}Y,Z) + g(\nabla_{X3}Y,Z)$$

$$g(\nabla_X Y) = g(\nabla_{X1}Y_1,Z) + g(\nabla_{X1}Y_2,Z) + g(\nabla_{X1}Y_3,Z) + g(\nabla_{X2}Y_1,Z) + g(\nabla_{X2}Y_2,Z) + g(\nabla_{X2}Y_3,Z) + g(\nabla_{X3}Y_1,Z) + g(\nabla_{X3}Y_2,Z) + g(\nabla_{X3}Y_3,Z). \quad (2.1)$$

Applying the proposition (2.1) we get:

$$g(\nabla_X Y) = g_1(\nabla^1{}_{X1}Y_1,Z_1) + fX_1(f)g_2(Y_2,Z_2) + hX_1(h)g_3(Y_3,Z_3) + fY_1(f)g_2(X_2,Z_2) + f^2g_2(\nabla^2{}_{X2}Y_2,Z_2) - fZ_1(f)g_2(X_2,Y_2) + hX_2(h)g_3(Y_3,Z_3) + hY_1(h)g_3(X_3,Z_3) + hY_2(h)g_3(X_3,Z_3) + h^2g_3(\nabla^3{}_{X3}Y_3,Z_3) - h(Z_1+Z_2)(h)g_3(X_3,Y_3) \quad (2.2)$$

In the same way we have:

$$g(Y,\nabla_X Z) = g_1(Y_1,\nabla^1{}_{X1}Z_1) + fX_1(f)g_2(Y_2,Z_2) + hX_1(h)g_3(Y_3,Z_3) + fZ_1(f)g_2(X_2,Y_2) + f^2g_2(Y_2,\nabla^2{}_{X2}Z_2) - fY_1(f)g_2(X_2,Z_2) + hX_2(h)g_2(Y_3,Z_3) + hZ_1(h)g_3(X_3,Y_3) + hZ_2(h)g_3(X_3,Y_3) + h^2g_3(Y_3,\nabla^3{}_{X3}Z_3) - h(Y_1+Y_2)(h)g_3(X_3,Z_3) \quad (2.3)$$

In the same way we get again:

$$g(\nabla_Y X,Z) = g_1(\nabla_{Y1}X_1,Z_1) + fY_1(f)g_2(X_2,Z_2) + hY_1(h)g_3(X_3,Z_3) + fX_1(f)g_2(Y_2,Z_2) + f^2g_2(\nabla^2{}_{Y2},Z_2) - fZ_1(f)g_2(X_2,Y_2) + hY_2(h)g_3(X_3,Z_3) + hX_1(h)g_3(Y_3,Z_3) + hX_2(h)g_3(Y_3,Z_3) + h^2g_3(\nabla^3{}_Y X_3,Z_3) - h(Z_1+Z_2)(h)g_3(X_3,Y_3) \quad (2.4)$$

3

Again we have:

$$g(X,\nabla_Y Z) = g_1(X_1,\nabla_{Y1}Z_1) + fY_1(f)g_2(X_2,Z_2) + hY_1(h)g_3(X_3,Z_3) + fZ_1(f)g_2(X_2,Y_2) + f^2g_2(X_2,\nabla_{Y2}Z_2) - fX_1(f)g_2(Y_2,Z_2) + hY_2(h)g_3(X_3,Z_3) + hZ_1(h)g_3(X_3,Y_3) + hZ_2(h)g_3(X_3,Y_3) + h^2g_3(X_3,\nabla_{Y3}Z_3) - h(X_1+X_2)(h)g_3(Y_3,Z_3) \quad (2.5)$$

Then (2.5) + (2.6) gives:

$$g(\nabla_X Y,Z)+g(Y,\nabla_X Z) = g_1(\nabla_{X1}Y_1,Z_1)+g_1(Y_1,\nabla_{X1}Z_1)+f^2g_2(\nabla_{X2}Y_2,Z_2) + f2g2(Y2,\nabla X2Z2) +h2g3(\nabla X3Y3,Z3)+h2g3(Y3,\nabla X3Z3)+2fX_1(f)g_2(Y_2,Z_2) + 2hX_1(h)g_3(Y_3,Z_3)+ 2hX_2(h)g_3(Y_3,Z_3) \quad (2.6)$$

In the same way (2.7) + (2.8) gives:

$$g(\nabla_Y X,Z)+g(Y,\nabla_X Z) = g_1(\nabla_{Y1}X_1,Z_1)+g_1(X_1,\nabla_{Y1}Z_1)+f^2g_2(\nabla_{Y2}X_2,Z_2) + f2g2(X2,\nabla Y2Z2) + h2g3(\nabla Y3X3,Z3) + h2g3(X3,\nabla Y3Z3) + 2fY_1(f)g_2(X_2,Z_2) + 2hY_1g_3(X_3,Y_3) + 2hY_2g_3(X_3,Z_3) \quad (2.7)$$

on other hand we have:

$$X.g(Y,Z) = X_1.g(Y_1,Z) + X_1.g(Y_2,Z) + X_1.g(Y_3,Z)+ X_2.g(Y_1,Z) + X_2.g(Y_2,Z)X_2.g(Y_3,Z) + X_3.g(Y_1,Z) + X_3.g(Y_2,Z) + X_3.g(Y_3,Z) \quad (2.8)$$

so

$$g(Y,Z) = X_1.g(Y_1,Z_1) + f^2X_2g_2(Y_2,Z_2)++ 2fX_1(f)g_2(Y_2,Z_2) + 2hX_1(h)g_3(Y_3,Z_3) + 2hX_2(h)g_3(Y_3,Z_3) + h^2X_3g_3(Y_3,Z_3)$$

in th same way

$$g(X,Z)=Y_1.g(X_1,Z_1)+f^2g_2(X_2,Z_2)+h^2g_3(X_3,Z_3)+2fY_1(f)g_2(X_2,Z_2) +2hY_1(h)g_3(X_3,Z_3)+ 2hY_2(h)g_3(X_3,Z_3)$$

$$Y.g(X,Z) - (g(\nabla_Y X,Z) + g(X,\nabla_Y Z)) = (Y_1g_1(X_1,Z_1) - g_1(\nabla_{Y1}X_1,Z_1) - g1(X1,\nabla Z1Y1) + f2(Y2g2(X2,Z2) - g2(\nabla Y2X2,Z2) - g2(X2,\nabla Y2Z2)) + h2(Y3g3(X3,Z3) - g3(\nabla Y3X3,Z3) - g3(X3,\nabla Y3Z3))$$

As $(M_i,g_i,\nabla_i)$ is statistical with $i = 1,2,3$.Then we get

$$\overline{\nabla}_X g(Y,Z) = \nabla_Y g(X,Z)$$

Conversely if $(\overline{M},g,\overline{\nabla})$ is statistical it is clear that $(M_i,g_i,\nabla_i)$ is statistical with $i = 1,2,3$. □

Example. Let $M = (R \times_f R) \times_h R$ be a Riemannian manifold with $g = de_1^2 + f^2de_2^2 + h^2de_3^2$ and $\nabla$ an affine connection defined by :

$$\nabla_{e_1}e_1 = be_1, \nabla_{e_2}e_2 = \tfrac{}{2}e_1, \nabla_{e_3}e_3 = \tfrac{}{2}e_1 \quad bb$$
$$\nabla_{e_1}e_2 = \nabla_{e_2}e_1 = \tfrac{b}{2}e_2, \nabla_{e_1}e_3 = \nabla_{e_3}e_1 = \tfrac{b}{2}e_3, \nabla_{e_3}e_2 = \nabla_{e_2}e_3 = 0$$

where $\{e_1,e_2,e_3\}$ is an orthonormal led and $b$ is a constant.Then, $(M, g, \nabla)$ is a statistical sequential warped product manifold if $f = h = 1$.

**Proposition 2.5.** *Let $\overline{M} = (I \times_f M_2) \times_h M_3$ be a sequential generalized Robertson-Walker space -time with metric $g = (-dt^2 + f^2g_2) + h^2g_3$ and also let $X_i, Y_i \in \chi(M_i)$ for any $i = 2,3$. Then*

(1) $\overline{\nabla}_{\partial t}\partial t = 0$;

(2) $\overline{\nabla}_{\partial t}X_i = \overline{\nabla}_{X_i}\partial t = \frac{\dot f}{f}X_i, i = 2,3$;

(3) $\overline{\nabla}_{X_2}Y_2 = \nabla^2_{X_2}Y_2 - f\dot f g_2(X_2,Y_2)\partial t$;

(4) $\overline{\nabla}_{X_3}X_2 = \overline{\nabla}_{X_2}X_3 = X_2(\ln h)X_3$;

(6) $\overline{\nabla}_{X_3}Y_3 = \nabla^3_{X_3}Y_3 - hg_3(X_3,Y_3)\mathrm{grad}h$.

**Corollary 2.1.** $\overline{M} = (I \times_f M_2)\times_h M_3$ be a sequential generalized Robertson-Walker space -time.Assume that

*(1) $(M_i, \nabla_i, g_i)$ statistical manifold with $i = 2,3$,*

*(2) $f = h$,*

*(3) $Y_1g_2(X_2,Y_2) = X_1g_2(Y_2,Z_2)$.*

*Then $(\overline{M} = (I \times_f M_2) \times_h M_3, \overline{\nabla},g)$ is statistical.*

**Proposition 2.6.** *Let $\overline{M} = (M_1 \times_f M_2)\times_h I$ be a sequential standard static space -time with metric $g = (g_1+f^2g_2)+h^2(-dt^2)$ and also let $X_i,Y_i \in \chi(M_i)$ for any $i = 2,3$. Then*

(1) $\overline{\nabla}_{X_1}Y_1 = \nabla^1_{X_1}Y_1$;

(2) $\overline{\nabla}_{X_1}X_2 = \overline{\nabla}_{X_2}X_1 = X_1(\ln f)$;

(3) $\overline{\nabla}_{X_2}Y_2 = \nabla^2_{X_2}Y_2 - fg_2(X_2,Y_2)\mathrm{grad}^1 f$;

(4) $\overline{\nabla}_{X_i}\partial t = \overline{\nabla}_{\partial t}X_3 = X_i(\ln h)\partial t, i = 1,2$;

(5) $\overline{\nabla}_{\partial t}\partial t = h\,\mathrm{grad}h$.

**Corollary 2.2.** *$\overline{M} = (M_1 \times_f M_2) \times_h I$ a sequential standard static space -time.Assume that*

(1) *$(M_i,\nabla_i,g_i)$ statistical manifold with $i = 1,2$,*

(2) *$h$ a constant function on $M_1 \times_f M_2$.*

*Then $(\overline{M} = (I \times_f M_2) \times_h M_3, \overline{\nabla},g)$ is statistical.*

**Proposition 2.7.** *[3] Let $\overline{M} = (M_1 \times_f M_2) \times_h M_3$. be a sequential warped product manifold with metric $g = (g_1 \oplus f^2g_2) \oplus h^2g_3$ and also let $X_i,Y_i,Z_i \in \chi(M_i)$ for $i = 1,2,3$. Then*

(1) $\overline{R}(X_1,Y_1)Z_1 = R^1(X_1,Y_1)Z_1$;

(2) $\overline{R}(X_2,Y_2)Z_2 = R^2(X_2,Y_2)Z_2 - ||\mathrm{grad}^1||^2 [g_2(X_2,Z_2)Y_2 - g_2(Y_2,Z_2)X_2]$;

International Journal for Multidisciplinary Research (IJFMR)
E-ISSN: 2582-2160 ● Website: www.ijfmr.com ● Email: editor@ijfmr.com

(3) $\bar{R}(X_1,Y_2)Z_1 = \frac{-1}{f}H^f(X_1,Z_1)Y_2$;

(4) $\bar{R}(X_1,Y_2)Z_2 = fg_2(Y_2,Z_2)\nabla^1_{X_1}grad^1 f$;

(5) $\bar{R}(X_1,Y_2)Z_3 = 0$;

(6) $\bar{R}(X_i,Y_i)Z_j = 0, i \neq j$;

(7) $\bar{R}(X_i,Y_3)Z_j = \frac{-1}{h}H^h(X_i,Z_j)Y_2, i,j = 1,2$

(8) $\bar{R}(X_i,Y_3)Z_3 = hg_3(Y_3,Z_3)\nabla_{X_1}gradh, i = 1,2$

(9) $\bar{R}(X_3,Y_3)Z_3 = R^3(X_2,Y_3)Z_3 - ||gradh||^2\,[g_3(X_3,Z_3)Y_3 - g_3(Y_3,Z_3)X_3]$.

*The sequential warped product space $(M,\bar{g},\bar{\nabla},\bar{\nabla}^*)$ is dually at if and only if $(M_i,g_i,\nabla^i,\nabla^{i*})$ is a dually at space, when f and h are constant*

*warping functions, for i = 1,2,3. M is a proper sequential warped product manifold if $X_1(\ln f) \neq 0, X_1(\ln h) \neq 0$, and $X_2(\ln h) \neq 0$).*

***Corollary 2.3***. *If The statistical sequential warped product space $(M,\bar{g},\bar{\nabla},\bar{\nabla}^*)$ is dually at space then, $(M_i,g_i,\nabla^i,\nabla^{i*})$ is also dually at and $(M_i,g_i)$ is a Riemannian manifolds of constant sectional curvature. For i = 2,3.*

Proof. As the statistical sequential warped product space $(M, g, \nabla, \bar{\nabla}^*)$ is dually at space, from relation (1)of proposition (3.6), we get:
$R^1(X_1,Y_1)Z_1 = 0$ that implies ∇ is at thus $(M,g,\nabla,\nabla^*)$ is dually at.

From relations (2) and (9) of the same proposition, we have $R(X_2,Y_2)Z_2 = 0$ implies $R^2(X_2,Y_2)Z_2 = ||grad^1||^2\,[g_2(X_2,Z_2)Y_2 - g_2(Y_2,Z_2)X_2]$ with $c =$

$||grad^1 f||^2$. In the same way $R(X_3,Y_3)Z_3 = 0$ implies also $||gradh||^2$ with $c = ||gradh||^2$. □

**Corollary 2.4.** *Let $M = (M_1 \times_f M_2) \times_h I$ sequential warped product manifold*

*and $\bar{R}$ Riemannian curvature tensor on M. If X, Y, Z tangent to leaves. Then $\bar{R}(X, Y) Z$ is horizontal.*

**References**

1. Amari, S.,*Differential-Geometry methods in statsticals Lectures Notes in statistics*. Springer, Berlin. 30 (1985).
2. Amari,S. and Nagaoka,H., *Methods of information geometry*. Amer.Math.Soc.,Oxford University Press. Vol.191(2000).
3. De, U.C., Shenawy,S.,and Unaly,B., *Sequential warped products: curvature and conformal vector fields*. arXiv:1506.066056v2.(2019).
4. Diallo, A.S.,*Dualistic structures on doubly warped product manifolds*. Int. Elect. J. Geom.. 6(1) ,41-45 (2013).
5. Diallo, A.S and Todjihounde, *L.,Dualistic structures on twisted product manifolds*. Global Journal of Advanced Research on Classical and Modern Geometries.Vol.4, 35-43 (2015).
6. Djebbouri, D., and Ouakkas,S.,*Product of statistical manifolfd with doubly warped product*. Gen.Math.Notes.vol.31 No 2,16-28 (2015).
7. O'Neill, B., *Semi-Riemannian geometry*. Academic Press, New-York. (1983.)
8. Shenawy,S., *Note on sequential warped products:curvature and conformal vector fields*. arXiv:1506.06056v1. (2015.)

9. Todjihounde, L.,*Dualistic structures on warped product manifolds*. Di .Geom.-Dyn.Syst... 8, 278-284 (2006).
10. Unal,B.,*Multiply warped products.* J.Geom.Phys.. 34(3), 287-301 (2000).