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Linear Algebra: Eigen Values and Eigen Vectors and their Applications

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Abstract:

Linear algebra is the branch of mathematics that concerned with the study of vectors, vector spaces (Linear spaces), linear maps (linear transformations) and system of linear equations. Linear space is the central theme in modern mathematics that is widely used in both abstract algebra and functional analysis. Linear algebra also has a concrete representation in analytical geometry and in generalized operator theory. It has extensive applications in natural sciences as well as social sciences. Here in this paper we are presenting a study on linear algebra and associated linear equations, also the Eigen values and Eigen vectors as a subtopic and few applications of it. The work basically focuses on Eigen values and Eigen vectors their useful properties including some applications of it.

Keywords: Linear Algebra, Linear space, linear transformations, Eigen value, Eigen vector.

Introduction:

Linear algebra basically concern with the study of vectors, vector spaces (also called linear spaces) linear maps (also called linear transformations) as a branch of mathematics. A vector here is a directed line segment, characterized by both its magnitude represented by length and its direction. Vectors are used to represent physical entities such as force, velocity etc. they can be added to each other and multiplied by scalar quantity. A vector is also identified as a row or column matrix. A vector of dimension n is called n-space. Here in this paper we will introduce the idea of linear algebra, especially on matrix form of system of linear equations and related Eigen values and Eigen vectors of the matrix and some of their useful properties including few applications of them in natural sciences as well as in social sciences. In economics we frequently use the idea of matrices. Say 6-dimentional vectors i.e. of 6-tuples to represent Gross National product of six ASIAN countries. One can decide to display the GNP of six countries for a particular year where countries order is specified. For example (China, Japan, India, Indonesia, Bangladesh, Pakistan) by using vector $(v_1,v_2,v_3,v_4,v_5,v_6)$ where each countries GNP is in its respective position. A vector space (or linear space) as a purely abstract concept about which theorems are proved, is a part of abstract algebra and well integrated into this discipline. Some striking examples of these are the group of invertible linear maps or matrices and the ring of linear maps of a vector space. In the abstract setting of vector space the scalars with which an element of a vector space can multiplied are from a mathematical structure called **field**. This field usually the field of reals or field of complex numbers. Linear map takes elements from a linear space to another or to itself in a manner that is compatible with the addition and scalar multiplication given on the vector spaces. The set of such transformations is itself a vector space. The study of properties and algorithm on matrices including determinants, Eigen values and Eigen vectors are considered to be part of linear algebra. We simply say that linear problems of mathematics are those that exhibit linearity in their behavior. As an example in differential calculus we deal with linear approximation of functions. In many physical problems the study of various nonlinear situations is very important. So approximating the nonlinear problems to linear problem we can overcome those situations. Similar to these the concept of Eigen values and Eigen vectors are very important for intrinsic study of linear problems recurrently in many areas of science and technology. Among their many applications we have the inference of the type of extrema of a multivariate function, characterization of the stability and convergence of dynamical system, study of physical properties such as axes of inertia, characterization of human face etc.Here we aimed at addressing applications from the Eigen vector and Eigen value perspective, in the sense that the respective concept and some important properties are presented as a preparation for discussing some of the many representative applications of Eigen values and Eigen vectors. Eigen value and Eigen vectors are useful throughout pure and applied mathematics and they appear in settings far more general than we consider. Eigen values are also used to study differential equations and continuous dynamical system, they provide critical information in engineering design and they arise naturally in fields such as physics and chemistry.

Linear algebra:

Linear algebra is a branch of mathematics concerned with the study of vectors, vector spaces (or linear spaces), linear maps (or linear transformations) and system of linear equations. Vectors are the central theme in modern mathematics a vector here is a directed line segment characterized by both its magnitude represented by length and its direction. The zero vector is an exception. It has zero magnitude but no direction. Vectors are used to represent physical entities such as velocity, force etc. and can be added to each other and multiplied by scalars, thus forming first example of real vector space. Modern algebra has been extended to consider spaces of arbitrary or infinite dimension. A vector space of dimension n is also called an n-space. Although we cannot easily visualize vectors in n-space, such vectors or n-tuples are useful in representing data. These n-tuple vectors being consist of n-ordered components data can be efficiently summarized and manipulated in this framework.

Linear equations and Matrix representation*:*

A linear equation is an algebraic equation in which each term is either a constant or the product of a constant and a single variable. Linear equations can have one or more variables. Linear equations occur most frequently in most subareas of mathematics, especially in applied mathematics. Many non linear equations may also be reduced to linear equations by assuming that quantities of interest. Linear equations don't include exponents. A system of linear equations can be modeled to matrix equations. As for example linear system-

$a_{11}x_1+a_{12}x_2+a_{13}x_3+\dots\dots a_{1n}x_n=b_1$

$a_{21}x_2+a_{22}x_2+a_{23}x_3+\dots\dots a_{2n}x_n=b_2$

$\dots\dots\dots\dots$

$a_{n1}x_1+a_{n2}x_2+a_{n3}x_3+\dots\dots a_{nn}x_n=b_n$

can be written as Ax=b

Where $A=\begin{bmatrix} a_{11}\, a_{12} & .. & a_{1n} \\ a_{21}\, a_{22} & .. & a_{2n} \\ a_{n1}\ a_{n2} & \cdots & a_{nn} \end{bmatrix}$, $x=\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}$, $b=\begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix}$

For a square matrix the determinant and inverse matrix governs the behavior of solutions to the corresponding system of linear equations and the Eigen value and Eigen vector of allied matrix provide insight into the geometry of the associated linear transformations.

Let us consider n-dimensional vector space V and an m-dimensional vector space W with bases $B=\{v_1, v_2 \ldots\ldots\ldots v_n\}$ and $C=\{w_1, w_2 \ldots\ldots\ldots w_m\}$ respectively , then there is a bijective linear map from L(V,W) to $M_{\mathbf{nxm}}(R)$. Corresponding to this linear map we have

T $(v_i) \in W$ for which

T $(v_1) = a_{11}w_1 + a_{12}w_2 + \ldots\ldots\ldots\ldots a_{1n}w_n$

T $(v_2) = a_{21}w_1 + a_{22}w_2 + \ldots\ldots\ldots\ldots a_{2n}w_n$

………….. …………………………………… (1.1)

T $(v_n) = a_{n1}w_1 + a_{n2}w_2 + \ldots\ldots\ldots\ldots a_{nn}w_n$

Then we write matrix $M^v{}_w$ (T) of T with the choice of bases $\{v_i\}$ of V and $\{w_j\}$ of W to be transpose of the matrix of co efficient of equations (1.1). That is

$M^v{}_w\ (T) = \begin{bmatrix} a_{11}\, a_{12} & .. & a_{1n} \\ a_{21}\, a_{22} & .. & a_{2n} \\ a_{n1}\ a_{n2} & \cdots & a_{nn} \end{bmatrix}$

The matrix $M^v{}_w$ (T) is called matrix associated with T with respect to bases B and C, also $M^v{}_w$(T) is called the matrix representation of T with respect to the bases. The matrix $M^v{}_w$(T) is the **nxm** matrix when $\mathbf{i^{th}}$ column is coefficients of $T(v_i)$ when expressed as a linear combination of w_j , $1 \le i \le m$

As for example if we consider linear map T: $R^2 \to R^2$ BY $(x,y) \to (x+y, x-y)$ then using standard basis $\{e_1=(1,0), e_2=(0,1)\}$ for the space we have $Te_1=(1,1)=(1,0)+(0,1)=1.e_1+1.e_1$

$Te_2= (1,-1) =(1,0)+(0,-1)= 1.e_1-1.e_2$

So that we can have $M^v{}_w\ (T) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Eigen value and Eigen vector of a matrix:

Here in this paper I will focus on square matrices with respective applications that can be interpreted as transformations of the R^n space. The application $x \to Ax$ can transform a given vector x to another vector of different direction and length. However there may be some very special vectors for that transformation. As for example the set of vectors that are transformed to zero vector by the applications that is $\{x: Ax=0\}$. As much as we change the base the null space always same.

An Eigen vector of an **nxn** matrix A is a vector $x \in R^n$ other than zero, such that for any scalar $\tau \in R$ we have $Ax= \tau x$ this scalar τ is called Eigen value of A crrespnding to Eigen vector x. Thus a scalar τ is Eigen value of A corresponding to Eigen vector x if there is a nontrivial solution of $Ax= \tau x$.

For example, consider

A=$\begin{bmatrix}1 & 6\\5 & 2\end{bmatrix}$, u=$\begin{bmatrix}6\\-2\end{bmatrix}$ then u is Eigen vector of A.

For Au=$\begin{bmatrix}1 & 6\\5 & 2\end{bmatrix}\begin{bmatrix}6\\-2\end{bmatrix}$= - 4$\begin{bmatrix}6\\-2\end{bmatrix}$

For a given matrix A and a scalar τ the term det (A- τI)

gives a polynomial and the corresponding equation det (A- τI) =0 gives roots of τ, which we call characteristic root (or Eigen value). The possible way of obtaining the respective Eigen vector is by the substitution of the Eigen values one by one in the characteristic equation and selecting the respective linear system. For example suppose to find the Eigen value and Eigen vector of A=$\begin{bmatrix}5 & 0\\2 & 1\end{bmatrix}$

Here we consider characteristic equation det (A- τI) =0 ie $\begin{vmatrix}5-\tau & 0\\2 & 1-\tau\end{vmatrix}$=0

Which gives $(\tau-5)(\tau-1)$=0 i.e. $\tau = 5\ and\ 1$

Corresponding to $\tau = 1$ we have Eigen vector v_1=$\begin{bmatrix}0\\1\end{bmatrix}$

Corresponding to $\tau = 5$ we have Eigen vector v_2=$\begin{bmatrix}2\\1\end{bmatrix}$

Another consequence of testing whether a scalar $\tau = 7$ is Eigen value of a matrix A=$\begin{bmatrix}1 & 6\\5 & 2\end{bmatrix}$

We will consider equation Ax= $7x$, which gives homogenous equation (A-7I) x=0 and we check for nontrivial solution of the system.

Thus we have A-7I=$\begin{bmatrix}-6 & 6\\5 & -5\end{bmatrix}\sim\begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}$ from this it is clear that homogenous system has free variables.

Hence 7 is an Eigen value of the given matrix A. In fact the solution in parametric vector form x=$\begin{bmatrix}1\\1\end{bmatrix}x_1$,$x_1\neq 0$ are infinite vectors associated with the Eigen value 7.

Given an **nxn** matrix A, one of its Eigen value τ_i may corresponds to a multiple root of the characteristic equation implying that Eigen value to have respective algebraic multiplicity$\mu_A(\tau_i)$. An Eigen value τ_i can also have a respective geometric multiplicity $\gamma_A(\tau_i)$ which is given by $\gamma_A(\tau_i) = n - rank(A-\tau_i I)$. It can be shown that $1 \leq \gamma_A(\tau_i) \leq \mu_A(\tau_I) \leq n$. The $rank(A-\tau_i I)$>1 implying that there will be two or more linear independent Eigen vectors with the same Eigen value ,the respective τ_i are said to be degenerate. In case Eigen values have multiplicity 1, the spectrum is said to be simple. It is to be noted that set of Eigen values of a matrix is the spectrum of the matrix. Similar to the context of real Eigen value and Eigen vector we can also have complex Eigen value and complex Eigen vector.

Diagonalization of a matrix:

A square matrix A over a field is said to be diagonalizable if there exist an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. Thus matrix A is diagonalizable if and only if A is similar to a diagonal matrix D. Thus if A=PDP^{-1} for invertible matrix P then AP=PD and if v_i is column i of P then this matrix equally implies that $Av_i=\tau_i v_i$ that is the columns of P are Eigen vectors. P being invertible Colum vectors are linearly independent. If a matrix doesn't have linearly independent vector, it cannot be diagonalized.

As for example suppose we have to diagonalize the matrix A=$\begin{bmatrix}4 & 2\\3 & -1\end{bmatrix}$

Clearly from characteristic equation det (A- τI) =0, we have characteristic values as $\tau = 5, -2$

Corresponding Eigen vectors are $v_1=\begin{bmatrix}2\\1\end{bmatrix}$ and $v_2=\begin{bmatrix}-1\\3\end{bmatrix}$ so that $P=\begin{bmatrix}2 & -1\\1 & 3\end{bmatrix}$. Also $D=\begin{bmatrix}5 & 0\\0 & -2\end{bmatrix}$

It can be checked that AP=PD. i.e. $A=PDP^{-1}$ Thus A is diagonalizable.

Some useful properties:

1. Every vector space has a basis
2. Any two bases of the same vector space have same cardinality.
3. For a basis B= {b_1, b_2, …. b_n} of a vector space V, the coordinate mapping $x \to [x]_B$is a one to one linear transformation.
4. If vector space V is of dimension p , p ≥ 1, then any linearly independent set of exactly p elements in V is automatically a basis for V.
5. A square matrix A is invertible if and only if its determinants are non zero.
6. The trace of a diagonal matrix A corresponds to the sum of diagonal elements.
7. A square matrix A is orthogonal if and only if $AA^T=A^TA=I$
8. An **nxn** matrix A is diagonalizable if and only if A has n –linearly independent Eigen vectors.
9. If nxn matrix A has n-distinct Eigen value then it is diagonalizable.

Applications:

Example (1): **Use in the field of agriculture**.

Let us consider a practical example in the field of agriculture. A farmer owns land where he grows strawberries. Part of the strawberries is used in three different sectors. Part **A** for the production of cake, part **B** for production of Jam and the last part **C** in the local market. Part **A** produces revenue equal to four times the expenditure made for the strawberries; part **B** produces revenue of double the expenditure while part **C** produces revenue equal to two thirds, since it is based only on offers made during the fair. Our aim is to obtain revenue proportional to the money invested in strawberry seedling, whereby with the revenues other strawberries are grown which are redistributed equally in the three sectors. If we suppose that the firm initially invested Rs 4200 in seedlings let us see how much does it get after a year in the optimal situation?

Let us consider x_1,x_2 and x_3 as the investments in sectors A, B and C respectively. $\begin{matrix}A\\B\\C\end{matrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \to$

$$after\ a\ year\begin{pmatrix}4x_1\\2x_2\\\frac{2}{3}x_3\end{pmatrix}=\begin{pmatrix}x_1+3x_2\\x_2+x_2\\\frac{2}{3}x_3\end{pmatrix}\to redistribution\begin{pmatrix}x_1+\frac{1}{3}x_2+x_1\\x_2+\frac{1}{3}x_2+x_1\\\frac{2}{3}x_3+\frac{1}{3}x_2+x_1\end{pmatrix}=\begin{pmatrix}2x_1+\frac{1}{3}x_2\\x_1+\frac{4}{3}x_2\\x_1+\frac{1}{3}x_2+\frac{2}{3}x_3\end{pmatrix}$$

So the transformation of investment after one year is given by flowing map.

$f:R^3 \to R^3$ with $f(x_1,x_2, x_3)=(2x_1+\frac{1}{3}x_2, x_1+\frac{4}{3}x_2, x_1+\frac{1}{3}x_2+\frac{2}{3}x_3)$

Then the matrix associated with the linear map f with respect to the canonical bases in the domain and co domain is M=M_{cc} (f) = $\begin{pmatrix} 2 & \frac{1}{3} & 0 \\ 1 & \frac{4}{3} & 0 \\ 1 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$

Now suppose X be the vector of the initial distribution, we would like the initial distribution after one year to be of the type τx,so we look for the Eigen values and Eigen vectors of M. Clearly characteristic equation is det (M-τI) =0

i.e. $\begin{vmatrix} 2-\tau & \frac{1}{3} & 0 \\ 1 & \frac{4}{3}-\tau & 0 \\ 1 & \frac{1}{3} & \frac{2}{3}-\tau \end{vmatrix}$=0

i.e. $(\frac{2}{3}-\tau)\left(\tau^2-\frac{10}{3}\tau+\frac{7}{3}\right)=0$ which gives $\tau=\frac{2}{3},1,\frac{7}{3}$

The Eigen space related to $\tau_1=\frac{2}{3}$ is V_{τ_1}= {(0,0,t): t∈ R};

Eigen space corresponding to $\tau_2=1$ is V_{τ_2}= {(-$\frac{1}{3}u$,u,0): u∈ R};

Eigen space related to $\tau_3=\frac{7}{3}$ is V_{τ_3}={(r,r, $\frac{4}{5}r$): r∈ R};

We now interpret the result in the light of our problem. The Eigen value $\tau_1=\frac{2}{3}$ is a possible solution to our problem: investing Rs 4200 in sector C, but this solution would not be a wise choice as every year the investment would decrease more and more. The Eigen value $\tau_2=1$ does not give any solution to our problem as the Eigen vectors $(-\frac{1}{3}u,u,0)$ have no interest as we can't invest a negative amount of money. Considering the Eigen value $\tau_3=\frac{7}{3}$ with the Eigen vector (r,r, $\frac{4}{5}r$) , if we invest Rs 4200 then r+r+ $\frac{4}{5}r$=Rs 4200 so that r=Rs 1500 and $\frac{4}{5}r$=Rs 1200 Thus the optimal solution is to invest Rs 1500 respectively in sectors A and B and Rs 1200 in sector C . After one year, the amount of money will increase and we will have Rs 3500 in sector A and B while Rs 2800 in sector C. This provides a simple modeling of a process that can also be applied in the industrial sphere in the production of goods and in the organization of resources.

Example (2).

Applications to differential equations:

In many applied problems, several quantities are varying continuously in time and they are related by a system of differential equations such as-

$x_1'= a_{11}x_1+a_{12}x_2+a_{13}x_3+ \ldots\ldots\ldots\ldots+a_{1n}x_n$

$x_2'=a_{21}x_1+a_{22}x_2+a_{23}x_3+ \ldots\ldots\ldots\ldots+a_{2n}x_n$

…….. ……. ……….. . …….. ……

$x_n'=a_{n1}x_1+a_{n2}x_2+a_{n3}x_3+ \ldots\ \ldots\ldots +a_{nn}x_n$

here $x_1,x_2,x_3,\ \ldots x_n$ are differentiable functions of t, with derivatives $x_x',x_2',\ \ldots\ x_n'$ and the a_{ij} are constants. The crucial factor of this system is that it is linear. In matrix form above can be expressed as –

$x'=Ax$ ………………………………………………………………… (1.2)

Where $x(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \\ . \\ . \\ . \\ x_n(t) \end{bmatrix}$ $x(t)^{/}=\begin{bmatrix} x_1(t)^{/} \\ x_2(t)^{/} \\ . \\ . \\ . \\ x_n(t)^{/} \end{bmatrix}$ $A=\begin{bmatrix} a_{11}\ a_{12} & .. & a_{1n} \\ a_{21}\ a_{22} & .. & a_{2n} \\ a_{n1}\ \ a_{n2} & \cdots & a_{nn} \end{bmatrix}$

A solution of (1.2) is a vector valued function that satisfies (1.2) for all t in some interval of real numbers such as $t \geq 0$. Here equation (1.2) is linear because both differentiation of functions and multiplication of vectors by a matrix are linear transformations. Thus if u and v are solutions of $x^{/}=Ax$ then cu+dv is also a solution, because

$(cu+dv)^{/}=cu^{/}+dv^{/}= cAu+dAv= A(cu+dv)$

Also identically zero function is a trivial solution of (1.2). Thus the set of all solution of (1.1) is a subspace of the set of all continuous functions with values in R^n . It should be noted that there always exist a fundamental set of solutions to (1.2).If A is an **nxn** matrix then there are n-linearly independent functions in a fundamental set and each solutions of (1.2) is a unique linear combination of these n-functions. That is the fundamental set of solutions is a basis for the set of all solutions of (1.2) , and the solution set is an n-dimensional vector space of functions. If a vector x_o is specified then the initial value problem is to construct the function x such that $x^{/}=Ax$ and $x(0)=x_o$. When A is diagonal matrix the solution of (1.2) can be produced by elementary calculus.

As for example, consider

$$\begin{bmatrix} x_1(t)^{/} \\ x_2(t)^{/} \end{bmatrix}=\begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (1.3)$$

i.e. $x_1(t)^{/} = 3x_1(t)$

$x_2(t)^{/}= -5x_2(t)$ ………………………… …………..(1.4)

Then system (1.3) is said to be decoupled because each derivative of the function depends only on the function itself, not on some combination or coupling of both $x_1(t)$ and $x_2(t)$. Solution of system (1.4) are $x_1(t)=C_1e^{3t}$ and $x_2(t)=C_2e^{-5t}$ for any constants C_1 and C_2 . Each solution of (1.3) can be written in the form $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}=\begin{bmatrix} C_1e^{3t} \\ C_2e^{-5t} \end{bmatrix}=C_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}e^{3t}+C_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}e^{-5t}$

This example suggest that for the general equation $x^{/}=Ax$ a solution might be a linear combination of functions of the form $x(t)=Ve^{\tau t}$ …………………………………… (1.5)

For some τ and some fixed nonzero vector v. it can be seen that $x'(t) = \tau ve^{\tau t}$

$$Ax(t) = Ave^{\tau t}$$

Since $e^{\tau t}$ is never zero, $x'(t)$ will equal to Ax (t) if and only if $\tau\, x = Ax$, that is if and only if τ is an Eigen value of A and v is a corresponding eigenvector. Thus each Eigen value-eigenvector pair provides a solution (1.5) of $x^{/}=Ax$. Such solutions are sometimes called Eigen functions of the differential equation. Thus Eigen functions provide the key for solving system of differential equations.

Conclusion:

The present work presented briefly and in an introductory manner the concept of linear algebra that is linear space, linear transformations,linear equations and associated matrix representation of linear system. Also at the same time the concept of Eigen values and Eigen vectors, their important properties and their theoretical and practical applications. Further study of Eigen values, Eigen vectors opens up new prospects in theoretical and applied research. It is hoped that the covered presentation may motivate the reader to probe further in this interesting area.

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