Laplace-Stieltjes Transform and its Asymptotic Expansion

M. K. Nikam

Associate Professor, Engineering Science Department, AISSMS, College of Engineering, Pune-01

Abstract
In this paper we will present the method of asymptotic expansion of Laplace-Stieltjes transform defined as $[f(x)] = m(t,z) = \int_0^{\infty} \frac{e^{-zx}}{1+tx} f(x)dx$. Asymptotic Transforms which comes to further improvement the extension of the Laplace-Stieltjes Transforms to all locally integrable function as proposed by Watson's Lemma. It gives us a way to obtain the complete asymptotic expansion of integrals on the real axis that have an exponential type of kernel.. To obtain the asymptotic expansion of the transform, we follow the technique of Handelsman and Blestein.

Keywords: Laplace transform, Stieltjes transform, Laplace-Stieltjes transform

AMS Subject Classification No.: 41A60, 44A10.

1. Introduction

One of the generalization of Laplace-Stieltjes transform given by Giona and Paterno [1] is defined by the equation $M[f(x)] = \int_0^{\infty} \frac{e^{-zx}}{1+tx} f(x)dx$ …(1)
where $0 < z < \infty$ and $|\arg t| < \pi$. It is also denoted as $M[f(x)] = m(t,z)$ and known as Laplace-Stieltjes transform.

2. Existence of Laplace-Stieltjes Transform

Theorem: If the function f (x) is piecewise continuous on every finite interval in the range $0 < x < \infty$, then a Laplace-Stieltjes transform of f(x) exists for all
$Re(z) > M, |\arg t| < \pi$.

3. Classical Properties of the Laplace-Stieltjes transform

3.1 Linear Property

The Laplace-Stieltjes transform is linear for every pair of function $f_1(x)$ and $f_2(x)$ and every pair of constant a_1 and a_2 ,

$$M[a_1 f_1(x) + a_2 f_2(x)] = a_1 M[f_1(x)] + a_2 M[f_2(x)] \quad \ldots(2)$$

3.2 Translation Property or Shifting Theorem

If $M[f(x)] = m(t, z)$, when $Re(z) > \alpha$, α is positive number then

$$M[e^{-ax} f(x)] = m(t, z+a) , \quad \ldots(3)$$

$Re(z) > \alpha + a$, a is positive number.

3.3 Change of Scale Property

If M[f(x)] = m(t, z) , then $M[f(ax)] = \frac{1}{a} m(\frac{t}{a}, \frac{z}{a})$, …(4)

where 'a' is non- zero constant.

3.4 Multiplication by x^n

If f(x) is piecewise continuous and locally integrable in the interval $0 < x < \infty$
and if M[f(x)] = m(t, z), then

$$M[x^n f(x)] = (-1)^n \frac{d}{dz^n} m(t,z), \quad n = 1, 2, 3, \ldots \quad ...(5)$$

3.4 Division by x^n

If f(x) is piecewise continuous and locally integrable in the interval $0 < x < \infty$ and if M[f(x)] = m(t, z), then

$$M\left[\frac{f(x)}{x^n}\right] = \int_z^\infty \int_z^\infty \text{.........} \int_z^\infty m(t,u)(du)^n \quad , n = 1, 2, 3,\ldots \quad \ldots(6)$$

proved that $\lim_{x \to 0} \int \frac{f(x)}{x} dx$, exists.

3.5 Laplace-Stieltjes Transform of the derivative of f (x)

If f(x) is piecewise continuous and locally integrable in the interval $0 < x < \infty$ and if M[f(x)] = m(t, z), then

$$M[f'(x)] = -f(0) + (z+t).m(t,z) + t^2 \frac{d}{dt} m(t,z), \quad ...(7)$$

where $f'(x)$ denotes derivative of f(x).

3.6 Inversion Theorem

Let f(x) be continuous and x^{-1}f(x) be locally integrable function in the interval
$0 < x < \infty$, and if M[f(x)] = m(t, z) , then

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} m(t,z) \cdot e^{zx} (1+tx) dz, \quad \ldots(8)$$

where 'c' is constant and Re(z) > 0 , $|\arg t| < \pi$.

4. Asymptotic Expansion

Asymptotic analysis gives the behavior of function at and near given points in their domains of definition. To describe the behavior of the function F(z) as $|z| \to \infty$ within a sector $a \le \operatorname{Re} z \le b$, it is in many cases sufficient to derive an expression of the form $F(z) = \phi(z)[1 + r(z)]$ … (9)
where $\phi(z)$ is a function of similar structure of F(z) and r(z) converges to zero as $|z| \to \infty$ within the given sector . Equation (9) is called the asymptotic representation of F (z) for large |z|.

4.1. Introduction and Preliminary Results

4.1.1 Asymptotic sequence

A sequence of functions φ_n as n→ 0 to ∞ is called an asymptotic sequence at z → z_0 from S if whenever n > m, we have $\varphi_n(z) = o(\varphi_m(z))$ as z → z_0 from S.

4.1.2 Watson's Lemma

Supposethat f∈$f \in L^1_{loc}([0,\infty), X)$ has anasymptotic expansion in terms of $\{t^n\}_{n\in N}$ as t→0+; i.e., suppose that $f(t) \sim \sum_{n=0}^{\infty} c_n t^n$, Then, for any u∈{f} we have that

$u(z) \sim \sum_{n=0}^{\infty} c_n \frac{n!}{z^{n+1}}$ as z→∞

Wong [6] and Saxena [5] have obtained the asymptotic expansion of Stieltjes transform and generalized Stieltjes transform. Also the asymptotic expansions of Stieltjes transform and Laplace transform are obtained by Widder [7].Also some authors are given the asymptotic expansion of some integral transformations by using the different methods. In this article we obtain the asymptotic expansion of Laplace –Stieltjes transform defined by Giona and Paterno [2],

$$m(t,z) = M[f(x)] = \int_0^{\infty} \frac{e^{-zx}}{1+tx} f(x)dx$$

where z is complex variable and $t \in \left(C \setminus (-\infty, 0]\right)$, and studied by Chaudhary and Nikam [3]. To obtain the asymptotic expansion of the transform(1), we follow the technique of Handelsman and Blestein [4].

The basic idea to get the asymptotic expansion of the integral transform $I(\lambda) = \int_0^{\infty} h(\lambda x) f(x) dx$ given by Handelsman and Blestein is as follows:

If $f(x) \approx \sum_{r=0}^{\infty} c_r x^{a_r} \quad x \to 0$

and $h(x) \approx \sum_{r=0}^{\infty} d_r x^{-b_r}$ as $x \to \infty$ for $b_r \neq a_r + 1$ for any r , then as $\lambda \to \infty$

$$I(\lambda) = \sum_{r=0}^{\infty} \lambda^{-b_r} d_r M[f; 1-b_r] + \sum_{r=0}^{\infty} \lambda^{-1-a_r} c_r M[h; 1+a_r] \qquad \ldots (10)$$

Where $M[f;1-b_r]$ denotes the Mellin transform at function $f(x)$ with parameter λ , the Mellin transform of kernel h(x) evaluated at λ .

5. Asymptotic expansion of Laplace –Stieltjes transform

Let $f(x) \approx \sum_{r=0}^{\infty} c_r x^{a_r}$ as $x \to \infty$

For $x = \frac{1}{y}$ in (1.2), it becomes

$$m(t,z) = \int_0^{\infty} \frac{e^{-z/y}}{y+t} \frac{1}{y} f\left(\frac{1}{y}\right) dy$$

$$= \int_0^\infty \frac{e^{-z/y}}{y+t}\phi(y)dy \qquad \ldots.(11)$$

Where

$$\phi(y) = \frac{1}{y} f\left(\frac{1}{y}\right) = \sum_{r=0}^{\infty} c_r y^{a_r - 1} \quad \text{as} \quad y \to 0 \qquad \ldots.(12)$$

From the equation (1), kernel is

$$k(t,z;y) = \frac{e^{-zy}}{1+ty}$$

Considering expansion of Laplace –Stieltjes transform w.r.t. t, for t = 1,

$$k(1,z;y) = \frac{e^{-zy}}{1+y}$$

$$k(1,z;y) \sim \frac{1}{1+y}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}z^r y^r$$

$$k(1,z;y) \sim \frac{1}{y}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}z^r y^r \quad \text{, as } y \to \infty$$

$$\sim \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}z^r y^{-(1-r)} \quad \text{, as } y \to \infty$$

$$\sim \sum_{r=0}^{\infty} d_r y^{-b_r} \quad \ldots.(13)$$

where $d_r = \frac{(-1)^r z^r}{r!}$ and $b_r = 1 - r$

Using result [1, 312(4)], Mellin transform of kernel $k(1,z;y) = \frac{e^{-zy}}{1+y}$ is given by

$$\therefore M\left[k(1,z;y)\right] = M\left[\frac{e^{-zy}}{1+y}\right]$$

$$= \left[\Gamma p \cdot e^z \Gamma(1-p,z)\right] \qquad \ldots.(14)$$

where Re z > 0 , Re p > 0 and $\Gamma(a,z)$ is an incomplete gamma function.

From the above discussion and by using Handelsman and Blestein technique, we arrive at following theorem;

Theorem: If

$$\phi(y) = \sum_{r=0}^{\infty} c_r y^{a_r - 1} \quad \text{as} \quad y \to 0 \ ,$$

$$k(1,z;y) \sim \sum_{r=0}^{\infty} d_r y^{-b_r} \quad \text{and} \quad a_r \neq 1 - r$$

then as $t \to \infty$

$$m(t,z)=\sum_{r=0}^{\infty}\frac{(-1)^{r}z^{r}}{r!}\cdot t^{-(1-r)}M\left[\phi,r\right]+\sum_{r=0}^{\infty}c_{r}\cdot t^{-a_{r}}e^{z}\Gamma a_{r}\Gamma(1-a_{r},z)$$

which gives asymptotic expansion of Laplace –Stieltjes transform.

References

1. Erdelyi A., Magnus W., Oberhettinger F. and Tricomi F.G.; Tables of Integral Transform, Vol. I, II, Bateman project, McGraw Hill, 1954.
2. Giona M. and Paterno O., Integral transforms of multifractal measures, Fractals in Engineering : From Theory to Industrial Applications, Edited by J.L.Vehel, Evelyne Lutton and C. Tricot, Springer London, 1997, ‘p.19-31.
3. Chaudhary M.S. and Nikam M.K., On Laplace-Stieltjes Transform. J. Indian Acad. of Math. Vol.29 (2), 2007. (To appear)
4. Handelsman R.A. and Blestein N., Asymptotic expansion of Integrals.Holt,Rinehartand Winston, New York,1975.
5. Saxena R.K. and Gupta N., On asymptotic expansion of generalized Stieltjes transform. The Mathematics Students, Vol.64 (1-4), 1995(51-56)
6. Wong R., Explicit error terms for asymptotic expansion of Mellin convolution. J. Math. Anal. Appl., Vol.72 (2), 740-756(1979)
7. Wider, D. V., The Laplace transform, Princeton University Press, Princeton (1946).