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Cubic Singular Moduli $x_{5/2}$ and Series for $1/\pi$

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Abstract. In this study, we give proof of a Ramanujan type series for $1/\pi$, which is derived from cubic singular moduli $x_{5/2}$.

Keywords: Eisenstein Series; Modular Equations; Cubic Singular Moduli.

1. Introduction

Let $(b)_0 = 1$ and, for a positive integer n,

$$(b)_n := b(b+1)(b+2)\dots(b+n-1),$$

and

$${}_{s+1}F_s(a_1,\dots,a_{s+1};b_1,\dots,b_s;x) := \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\cdots(a_{s+1})_n}{(b_1)_n(b_2)_n\cdots(b_s)_n}\frac{x^n}{n!}, |x|<1,$$ s is a non negative integer and $a_1,\dots,a_{s+1},b_1,\dots,b_s$ are complex numbers.

Ramanujan recorded 17 series for $1/\pi$ in his famous paper "Modular equations and approximations to π"[1], [2, pp. 36--38]. Again, at the beginning of section 14 of the same paper [1], [2, p. 37] Ramanujan wrote, "There are corresponding theories in which q is replaced by one or other of the functions"

$$q_r := q_r(x) := \exp\left(-\pi\csc(\pi/r)\frac{{}_2F_1(\frac{1}{r},\frac{r-1}{r};1;1-x)}{{}_2F_1(\frac{1}{r},\frac{r-1}{r};1;x)}\right), \quad (1.1)$$

where $r = 3$, 4, or 6. Here $r = 3$ corresponds to Ramanujan's theory of q_3, where

$$q_3 := q_3(x) := \exp\left(-\frac{2\pi}{\sqrt{3}}\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1-x)}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;x)}\right). \quad (1.2)$$

J.M. and P.B. Borwein [3] proved all these 17 series for $1/\pi$. See [4] for a current overview of Ramanujan's series for $1/\pi$.

2. Definitions, Preliminary Results and Notation

Following is the fundamental inversion formula recorded by Ramanujan [5, p. 258] in his second notebook.

$$z := z(q_3) := {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;x\right) = a(q_3), \quad (2.1)$$

where q_3 is given by (1.2) and

$$a(q_3) := \sum_{m,n=-\infty}^{\infty} q_3^{m^2+mn+n^2}. \quad (2.2)$$

Borwein brothers [6, p. 695, Theorem 2.3] were the first to prove this result in print. Later Berndt, Bhargava, and Garvan [7], [8, p. 99] also proved this result. In the remaining part of this paper, we will use $q = q_3$ and $x = x(q)$.

We now define a cubic modular equation of degree n. Suppose that the equality

$$n\frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-k^2\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; k^2\right)} = \frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-l^2\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; l^2\right)} \tag{2.3}$$

holds for some positive integer n. A modular equation of degree n is a relation between the moduli k and l that is given by (2.3). Ramanujan recorded his modular equations in terms of α and β, where $\alpha = k^2$ and $\beta = l^2$. β is said to have degree n over α. The corresponding multiplier m is defined by

$$m := m(\alpha, \beta) := \frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \alpha\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \beta\right)} = \frac{z(q)}{z(q^n)}. \tag{2.4}$$

We define Ramanujan's Eisenstein series

$$P(q) := 1 - 24\sum_{k=1}^{\infty} \frac{kq^k}{1-q^k}, \quad |q| < 1, \tag{2.5}$$

$P(q)$ satisfies the identity [7, Lemma 4.1]

$$P(q) = (1-4x)z^2 + 3H(q)z\frac{dz}{dx}, \tag{2.6}$$

where

$$H(q) = 4x(q)(1-x(q)).$$

. We set

$$x_n := x(e^{-2\pi\sqrt{n/3}}) \ and \ z_n := z(e^{-2\pi\sqrt{n/3}}). \tag{2.7}$$

The numbers x_n are cubic singular moduli. Also, it is shown that [9, Eqs. (3.11), (3.7) and (3.10)]

$$1 - x_n = x_{1/n}, \quad z_{1/n} = \sqrt{n}z_n, \quad and \quad m(x_{1/n}) = \sqrt{n}, \tag{2.8}$$

where $m = m(x(q), x(q^n))$ is the multiplier defined by (2.4). We end this section by stating the following lemmas.

Lemma 2.1 [3, p.178 proposition 5.6(b)] *Let*

$$C_k := \frac{\left(\frac{1}{3}\right)_k\left(\frac{1}{2}\right)_k\left(\frac{2}{3}\right)_k}{k!^3}$$

. If $z := z(3; q)$ is defined by (2.1), then

$$z^2 = {}_3F_2\left(\frac{1}{3}, \frac{2}{3}, \frac{1}{2}; 1,1; H\right) = \sum_{k=0}^{\infty} C_k H^k, \quad 0 \le x \le \frac{1}{2} \tag{2.9}$$

Lemma 2.2 [10, p. 367, Eq. (5.8)]

$$P(e^{-2\pi\sqrt{n/3}}) = \sum_{k=0}^{\infty} \{6(1-2x_n)k + 1 - 4x_n\}C_k H_n^k. \tag{2.10}$$

where $H_n = 4x_n(1-x_n)$.

Lemma 2.3 From H.H. Chan and W.-C. Liaw's paper [9, Eqs. (3.6) and (3.17)], we have,

$$nP(q^n) - P(\mathrm{q}) = \mathrm{z(q)z}(q^n)\left\{\left(1 - 4x(q^n)\frac{n}{m} - \left(1 - 4x(q)\right)m - 12x(q)(1 -\right.\right.$$

$x(q))\Big)\frac{dm}{dx(q)}\Big\}$ (2.11)

and

$$nP(e^{-2\pi\sqrt{n/3}}) + P(e^{-2\pi/\sqrt{3n}}) = \frac{6\sqrt{3n}}{\pi} - 2nz_n^2. \quad (2.12)$$

Three series for $1/\pi$ were derived by Baruah and Berndt [10] by substituting n = 2, 3 and 5 in (2.10)--(2.12). In the final section of this paper we derive a series for $1/\pi$ for n = 5/2.

3. Series corresponding to $n = 5/2$

Theorem 3.1 *If C_k, $k \geq 0$, is defined as above, then*

$$\frac{324\sqrt{15}}{\pi\sqrt{2}} = \sum_{k=0}^{\infty}\{30(35\sqrt{2} - 2\sqrt{5})k + 125\sqrt{2} + 16\sqrt{5}\}C_k\left(\frac{(223+70\sqrt{10})}{1458}\right)^k. \quad (3.1)$$

Proof. Setting $n = 5/2$ in (2.11), and replacing q by q^2 we arrive at

$$5P(q^5) - 2P(\mathrm{q}^2) = 2\mathrm{z}(\mathrm{q}^2)\mathrm{z}(q^5)\left\{\left(1 - 4x(q^5)\frac{5}{2\acute{m}} - (1 - 4x(q^2))\acute{m} - 12x(q^2)(1 - x(q^2))\right)\frac{d\acute{m}}{dx(q^2)}\right\} \quad (3.2)$$

where . $\acute{m} = \frac{\mathrm{z}(\mathrm{q}^2)}{\mathrm{z}(q^5)}$.

Again, setting $q = e^{-2\pi/\sqrt{30}}$ in (3.2) and employing (2.8), we obtain

$$5P\left(e^{-2\pi\sqrt{5/6}}\right) - 2P\left(e^{-2\pi\sqrt{2/15}}\right) = 2\sqrt{5/2}z_{5/2}{}^2\left\{\left(1 - 4x_{\frac{5}{2}}\right)\sqrt{5/2} - \left(4x_{\frac{5}{2}} - 3\right)\sqrt{\frac{5}{2}} - 12x_{\frac{5}{2}}(1 - x_{5/2})\left[\frac{d\acute{m}}{dx(q^2)}\right]_{q=e^{-2\pi/\sqrt{30}}}\right\} \quad (3.3)$$

To calculate $x_{5/2}$, we recall the following identity from [12, Eq. (2.7)]

For $q = e^{-2\pi\sqrt{n/3}}$, if

$$\mu_n = \frac{f^6(-q)}{3\sqrt{3q}f^6(-q^3)},$$

where $f(-q) = \prod_{k=1}^{\infty}(1 - q^k),$

then

$$\frac{1}{x_n} = \mu_n{}^2 + 1 \quad (3.4)$$

and $\mu_{1/n} = \frac{1}{\mu_n}$ (3.5)

The parameter μ_n was introduced by K.G. Ramanathan[15, Eq.(51)].

Next, we recall two further identities from [14, Theorem. 4.4], namely

$$3\left((\mu_n\mu_{25n})^{1/3} + \frac{1}{(\mu_n\mu_{25n})^{1/3}}\right) + 5 = \left(\frac{\mu_{25n}}{\mu_n}\right)^{1/2} - \left(\frac{\mu_n}{\mu_{25n}}\right)^{\frac{1}{2}}, \quad (3.6)$$

and

$$3\left((\mu_n\mu_{4n})^{1/3} + \frac{1}{(\mu_n\mu_{4n})^{1/3}}\right) = \frac{\mu_n}{\mu_{4n}} + \frac{\mu_{4n}}{\mu_n} \quad (3.7)$$

Setting $n = 1/10$ in (3.6) and (3.7) and using (3.5), we arrive at

$$3\left(A+\frac{1}{A}\right)+5=B^3-\frac{1}{B^3} \tag{3.8}$$

and

$$3\left(B^2+\frac{1}{B^2}\right)+5=A^3+\frac{1}{A^3}, \tag{3.9}$$

where $A=\left(\frac{\mu_{10}}{\mu_{5/2}}\right)^{1/3}$ and $B=\left(\mu_{10}\mu_{5/2}\right)^{1/6}$. (3.10)

Solving for A and B from (3.8) – (3.9), we obtain

$$\mu_{10}\mu_{5/2}=99+70\sqrt{2} \text{ and } \frac{\mu_{10}}{\mu_{5/2}}=9+4\sqrt{5} \tag{3.11}$$

Thus, from (3.11) and (3.4), we deduce that

$$x_{10}=\frac{1}{2}-\frac{35\sqrt{2}+2\sqrt{5}}{108} \text{ and } x_{5/2}=\frac{1}{2}-\frac{35\sqrt{2}-2\sqrt{5}}{108} \tag{3.12}$$

Next, we evaluate $\frac{d\acute{m}}{dx(q^2)}$ at $q=e^{-2\pi/\sqrt{30}}$. We have

$$\frac{d\acute{m}}{dx(q^2)}=\frac{d\acute{m}}{dx(q)}\frac{dx(q)}{dx(q^2)} \tag{3.13}$$

But, by theorem 2.3 of [11],

$$\frac{dx(q^n)}{dx(q)}=\frac{n}{m^2}\frac{x(q^n)(1-x(q^n))}{x(q)(1-x(q))} \tag{3.14}$$

where $m=\frac{z(q)}{z(q^n)}$

In Particular, when $n=2$,

$$\frac{dx(q^2)}{dx(q)}=\frac{2}{m^2}\frac{x(q^2)(1-x(q^2))}{x(q)(1-x(q))} \tag{3.15}$$

Setting $q=e^{-2\pi/\sqrt{30}}$ in (3.15), so that $(q)=x_{1/10}=1-x_{10}$ and $x(q^2)=x_{2/5}=1-x_{5/2}$, we obtain

$$\left[\frac{dx(q^2)}{dx(q)}\right]_{q=e^{-2\pi/\sqrt{30}}}=\frac{1}{2}\left(\frac{z_{5/2}}{z_{10}}\right)^2.\frac{x_{\frac{5}{2}}(1-x_{5/2})}{x_{10}(1-x_{10})} \tag{3.16}$$

To evaluate $\frac{z_{5/2}}{z_{10}}$, we recall from Theorem7.1(iii)[8, p. 120] that

$$m=\frac{z(q)}{z(q^2)}=\frac{(1-x(q^2))^{2/3}}{(1-x(q))^{1/3}}-\frac{x^{\frac{2}{3}}(q^2)}{x^{\frac{1}{3}}(q)} \tag{3.17}$$

Setting $q=e^{-2\pi/\sqrt{30}}$ in (3.17) and using (3.12), we arrive at

$$\frac{z_{5/2}}{z_{10}}=\frac{3\sqrt{2}}{\sqrt{10}+1} \tag{3.18}$$

Employing (3.18) and (3.12) in (3.16), we obtain

$$\left[\frac{dx(q^2)}{dx(q)}\right]_{q=e^{-2\pi/\sqrt{30}}}=\frac{5699+1802\sqrt{10}}{81} \tag{3.19}$$

Now, we evaluate $\left[\frac{d\acute{m}}{dx(q)}\right]_{q=e^{-2\pi/\sqrt{30}}}$. To this end, differentiating $\acute{m}=\frac{z(q^2)}{z(q^5)}$ with respect to $x=x(q)$, we obtain

$$\frac{d\acute{m}}{dx(q)}=\frac{z(q^2)}{z(q)}.\frac{d}{dx(q)}\frac{z(q)}{z(q^5)}+\frac{z(q)}{z(q^5)}\frac{d}{dx(q)}\frac{z(q^2)}{z(q)} \tag{3.20}$$

To evaluate $\frac{d}{dx(q)}\frac{z(q^2)}{z(q)}$, we recall the following modular equation of degree 2 from [8, p.120, Theorem 7.1]. If $\beta=x(q^2)$ has degree 2 over $\alpha=x(q)$, then

$$\frac{z(q^2)}{z(q)} = \frac{1}{2}\left\{\frac{(x(q))^{2/3}}{x(q^2)^{1/3}} - \frac{(1-x(q))^{2/3}}{(1-x(q^2))^{1/3}}\right\} \tag{3.21}$$

Differentiating (3.21) with respect to $x(q)$ and then setting $q = e^{-2\pi/\sqrt{30}}$, we arrive at

$$\left[\frac{d}{dx(q)}\frac{z(q^2)}{z(q)}\right]_{q=e^{-2\pi/\sqrt{30}}} = -\frac{2}{3}\left(13+4\sqrt{10}\right) \tag{3.22}$$

Now we evaluate $\left[\frac{d}{dx(q)}\frac{z(q)}{z(q^5)}\right]_{q=e^{-2\pi/\sqrt{30}}}$. To this end, differentiating (3.14) with respect to $x = x(q)$, we deduce that

$$m^2\frac{d^2(x(q^n))}{dx^2(q)} + 2m\frac{dm}{dx(q)}.\frac{dx(q^n)}{dx(q)} = n.\frac{x(q^n)(1-x(q^n))}{x(q)(1-x(q))}\left\{\left(\frac{1}{x(q^n)} - \frac{1}{1-x(q^n)}\right)\frac{dx(q^n)}{dx(q)} - \frac{1}{x(q)} + \frac{1}{1-x(q)}\right\} \tag{3.23}$$

For $n = 5$,

$$m^2\frac{d^2(x(q^5))}{dx^2(q)} + 2m\frac{dm}{dx(q)}.\frac{dx(q^5)}{dx(q)} = 5.\frac{x(q^5)(1-x(q^5))}{x(q)(1-x(q))}\left\{\left(\frac{1}{x(q^5)} - \frac{1}{1-x(q^5)}\right)\frac{dx(q^5)}{dx(q)} - \frac{1}{x(q)} + \frac{1}{1-x(q)}\right\} \tag{3.24}$$

where $m = \frac{z(q)}{z(q^5)}$.

To calculate $\frac{d^2(x(q^5))}{dx^2(q)}$, we recall from [8, p. 124] the following modular equations of degree 5 in the cubic theory. If $x(q^5)$ has degree 5 over $x(q)$, then

$$(x(q)x(q^5))^{1/3} + \{(1-x(q))(1-x(q^5))\}^{1/3} + 3\{x(q)x(q^5)(1-x(q))(1-x(q^5))\}^{1/6} = 1 \tag{3.25}$$

Differentiating (3.25) twice with respect to $x = x(q)$ and then setting $q = e^{-2\pi/\sqrt{30}}$ and using (3.12), (3.19), we deduce that

$$\left[\frac{d^2(x(q^5))}{dx^2(q)}\right]_{q=e^{-2\pi/\sqrt{30}}} = \frac{856(5699+1802\sqrt{10})}{9(85\sqrt{2}-53\sqrt{5})} \tag{3.26}$$

Again, setting $q = e^{-2\pi/\sqrt{30}}$ in (3.24) and (3.13) and using (3.19), we arrive at

$$\left[\frac{dm}{dx(q)}\right]_{q=e^{-2\pi/\sqrt{30}}} = \left[\frac{d}{dx(q)}\frac{z(q)}{z(q^5)}\right]_{q=e^{-2\pi/\sqrt{30}}} = \frac{1404}{223-70\sqrt{10}} \tag{3.27}$$

and

$$\left[\frac{d\acute{m}}{dx(q^2)}\right]_{q=e^{-2\pi/\sqrt{30}}} = \frac{666}{\sqrt{29917+9460\sqrt{10}}} \tag{3.28}$$

With the aid of (3.28) and (3.12) , we can rewrite (3.3)

$$5P\left(e^{-2\pi/\sqrt{5/6}}\right) - 2P\left(e^{-2\pi\sqrt{2/15}}\right) = \frac{25\sqrt{2}-4\sqrt{5}}{3}{z_{5/2}}^2 \tag{3.29}$$

Again, setting $n = 5/2$ in (2.12), we obtain

$$5P\left(e^{-2\pi/\sqrt{5/6}}\right) + 2P\left(e^{-2\pi\sqrt{2/15}}\right) = \frac{12\sqrt{15}}{\pi\sqrt{2}} - 10{z_{5/2}}^2 \tag{3.30}$$

Adding (3.29) and (3.30) and with the aid of (2.9), we arrive at

$$P\left(e^{-2\pi/\sqrt{5/6}}\right) = \frac{3\sqrt{6}}{\pi\sqrt{5}} + \left(\frac{25\sqrt{2}-4\sqrt{5}-30}{30}\right)\sum_{k=0}^{\infty} C_k {H_{5/2}}^k \tag{3.31}$$

Finally, setting $n = 5/2$ in (2.10) and with the aid of (3.12), we arrive at

$$P\left(e^{-2\pi/\sqrt{5/6}}\right) = \sum_{k=0}^{\infty}\left\{\left(\frac{35\sqrt{2}-2\sqrt{2}}{9}\right)k + \frac{1}{27}\left(-27+35\sqrt{2}-2\sqrt{5}\right)\right\}C_k {H_{5/2}}^k, \quad (3.32)$$

where $H_{5/2} = \frac{223+70\sqrt{10}}{1458}$.

From (3.31) and (3.32), we arrive at (3.1).

References

1. Ramanujan S., Modular equations and approximations to π, Quart. J. Math. (Oxford), **45** (1914), 350–372.
2. Ramanujan S., Collected Papers, Cambridge Univ. Press, Cambridge, 1927; reprinted by Chelsea, New York, *1962; reprinted by Amer. Math. Soc., Providence, RI, 2000*.
3. Borwein J. M., Borwein P. B., Pi and the AGM; A Study in Analytic Number Theory and Computational Complexity, Wiley, New York, 1987.
4. Baruah N. D., Berndt B. C. and Chan H. H. Ramanujan's series for $1/\pi$: A survey , American Mathematical Monthly, to appear.
5. Ramanujan S., Notebooks (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957.
6. Borwein J. M., Borwein P. B., A cubic counterpart of Jacobi's identity and the AGM, Trans. Amer. Math. Soc., **323** (1991), 691–701.
7. Berndt B. C., Bhargava S., Garvan F. G., Ramanujan's theories of elliptic functions to alternativebases, Trans. Amer. Math. Soc., **347** (1995), 4136–4244.
8. Berndt B. C., Ramanujan's Notebooks, Part V, Springer-Verlag, New York, 1998.
9. Chan H. H., Liaw W. -C., Tan V., Ramanujan's class invariant λ_n and a new class of series for $1/\pi$, London Math. Soc.(2), **64** (2001), 93–106.
10. Baruah N. D., Berndt B. C., Ramanujan's series for $1/\pi$ arising from his cubic and quartic theoriesof elliptic functions, J. Math. Anal. Appl., **341** (2008), 357–371.
11. Chan H. H., Liaw W. -C., Cubic modular equations and new Ramanujan-type series for $1/\pi$, Pacific J. Math **343** (2000), 219–238 .
12. Chan H. H., On Ramanujan's cubic transformation formula for ${}_2F_1(1/3,2/3;1;z)$, Math. Proc. Cambridge Philos. Soc., **124** (1998), 193–204 .
13. Ramanathan K. G., Some applications of Kronecker limit formula, J. Indian Math. Soc., **52** (1987),71–89.
14. Baruah N. D., Saikia N., Some new explicit values of ramanujan's continued fractions, Indian J. Math., **46** (2004), 357–371.
15. Ramanathan, K.G. Some applications of Kronecker limit formula, J. Indian Math. Soc. **52**, 71-89,1987