International Journal for Multidisciplinary Research (IJFMR)
E-ISSN: 2582-2160 ● Website: www.ijfmr.com ● Email: editor@ijfmr.com

Rogers-Ramanujan Type Identities Modulo 5, 7, 15 and 21.

Shaikh Fokor Uddin Ali Ahmed

Assistant Professor, Department Of Mathematics, F. A. Ahmed College, Garoimari, Kamrup, Assam, India.

Abstract:
In this paper, some identities of Rogers_Ramanujan Type related to modulo 5, 7, 15 and 21 is derived with the incorporation of generalized Bailey pairs and some standard results established by Andrew V. Sills [1] using some $q-$ difference relations.

Keywords: Rogers-Ramanujan Type Identities, Jacobi's Triple Product Identity, Bailey Pairs.

Mathematics Subject Classification: 11P84, 11P81, 33D15, 05A17

Introduction:
For |q|<1, the q-shifted factorial is defined by

$$(a;q)_0 = 1$$
$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \text{ for } n\geq 1$$
$$\text{and} \quad (a;q)_\infty = \prod_{k=1}^{\infty}(1-aq^k).$$

It follows that $(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$

The multiple q-shifted factorial is defined by

$$(a_1,a_2,\ldots,a_m;q)_n = (a_1;q)_n(a_2;q)_n\ldots(a_m;q)_n$$
$$(a_1,a_2,\ldots,a_m;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\ldots(a_m;q)_\infty.$$

The Basic Hyper geometric Series is

$${}_{p+1}\phi_{p+r}\begin{pmatrix} a_1,a_2,\ldots\ldots,a_{p+1};q;x \\ b_1,b_2,\ldots\ldots,b_{p+r} \end{pmatrix} = \sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n\ldots(a_{p+1};q)_n x^n(-1)^{nr}q^{\frac{n(n-1)r}{2}}}{(q;q)_n(b_1:q)_n(b_2;q)_n\ldots(b_{p+r};q)_n}$$

The series ${}_{p+1}\phi_{p+r}$ converges for all positive integers r and for all x. For $r=0$ it converges only when $|x|<1$.

Jacobi's Triple Product Identity:(see [5] 2.2.10 and 2.2.11)

$$(zq^{\frac{1}{2}}, z^{-1}q^{\frac{1}{2}}, q:q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n z^n q^{n^2/2} \tag{1.1}$$

And its corollary

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{(2k+1)\frac{n(n+1)}{2}-in} = \sum_{n=-\infty}^{\infty}(-1)^n q^{(2k+1)\frac{n(n+1)}{2}-in}(1-q^{(2n+1)i})$$

$$=\prod_{n=0}^{\infty}(1-q^{(2k+1)(n+1)})(1-q^{(2k+1)n+i})(1-q^{(2k+1)(n+1)-i}) \quad (1.2)$$

Definition 1: A pair of sequences $(\alpha_n(a,q),\beta_n(a,q))$ is called a Bailey pair if for $n\geq 0$,

$$\beta_n(a,q)=\sum_{r=0}^{n}\frac{\alpha_r(a,q)}{(q,q)_{n-r}(aq,q)_{n+r}} \quad (1.3)$$

In [4] and [5], Bailey proved the following result known as "Bailey Lemma".

Bailey's Lemma: If $(\alpha_r(a,q),\beta_j(a,q))$ form a Bailey pair, then

$$\frac{1}{(\frac{aq}{\rho_1};q)_n(\frac{aq}{\rho_2};q)_n}\sum_{j\geq 0}\frac{(\rho_1;q)_j(\rho_2;q)_j(\frac{aq}{\rho_1\rho_2};q)_{n-j}}{(q;q)_{n-j}}(\frac{aq}{\rho_1\rho_2})^j\beta_j(a;q)$$

$$=\sum_{r=0}^{n}\frac{(\rho_1;q)_r(\rho_2;q)_r}{(\frac{aq}{\rho_1};q)_r(\frac{aq}{\rho_2};q)_r(q;q)_{n-r}\,(aq;q)_{n+r}}(\frac{aq}{\rho_1\rho_2})^r\alpha_r(a;q) \quad (1.4)$$

Corollary: If $(\alpha_m(a,q),\beta_j(a,q))$ form a Bailey pair, then

$$\sum_{j\geq 0}a^jq^{j^2}\beta_j(a,q)=\frac{1}{(aq;q)_\infty}\sum_{m=0}^{\infty}a^mq^{m^2}\alpha_m(a,q) \quad (1.5)$$

In [4] and [5], Bailey considered several Bailey pairs which are special cases of a more general Bailey pair involving additional parameters d and k.

Parameterized Bailey pair:

Let $\lambda=-\frac{3}{2}d^2+dk+\frac{1}{2}d$, $h=|\frac{2\lambda}{d}|$, and $t=d+h+2$.

Let $\alpha_{d,k,m}(a,q)=\begin{cases}\frac{(-1)^ra^{(k-d)r}q^{\left(dk-d^2+\frac{d}{2}\right)r^2-\frac{d}{2}r}(aq^{2d};q^{2d})_r(a;q^d)_r}{(a;q^{2d})_r(q^d;q^d)_r} & \\ 0 & if\ m=dr, and\ otherwise\end{cases}$

and

$$\beta_{d,k,m}(a,q)=\begin{cases}\lim_{r\to 0}\frac{{}_{t+1}W_t\,(a;\gamma_1,\gamma_2,\ldots,\gamma_h,\mu_1,\mu_2,\ldots,\mu_d;q^d;\tau^ha^{k-d}q^{nd})}{(a,aq;q)_n} & if\ \lambda\geq 0\\ \lim_{r\to 0}\frac{{}_{t+1}W_t\,(a;\delta_1,\delta_2,\ldots,\delta_h,\mu_1,\mu_2,\ldots,\mu_d;q^d;\frac{a^{k-d}q^{nd}}{\tau^h})}{(a,aq;q)_n} & if\ \lambda<0\end{cases} \quad (1.6)$$

where $\gamma_j=\frac{q^{\lambda/h}}{\tau}$, $\mu_j=q^{d-j-n}$, $\delta_j=\tau aq^{d-\lambda/h}$.

$${}_{s+1}W_s\,(a_1;a_4,a_5,\ldots,a_{s+1};q;z)={}_{s+1}\phi_s\begin{bmatrix}a_1,qa_1^{1/2},-qa_1^{1/2},a_4,\ldots,a_{s+1};q,z\\ a_1^{1/2},-a_1^{1/2},\frac{qa_1}{a_4},\ldots,\frac{qa_1}{a_{s+1}}\end{bmatrix},$$

and,

$${}_{s+1}\phi_s\begin{bmatrix}a_1,a_2,\ldots,a_{s+1};q,z\\ b_1,b_2,\ldots,b_s\end{bmatrix}=\sum_{r=0}^{\infty}\frac{(a_1,a_2,\ldots,a_{s+1};q)_r}{(q,b_1,b_2,\ldots,b_s;q)_r}.\,z^r.$$

Then $\alpha_{d,k,m}(a,q)$ and $\beta_{d,k,m}(a,q)$ form a Bailey pair.

Bailey considered the special cases $\alpha_{d,k,m}(a,q)$ for $(d,k)=(1,2),(2,2),(2,3)\,and\,(3,4)$ in [5]. Each of these four (d,k) sets is particularly nice, as the resulting expression for $\alpha_{d,k,m}(a,q)$ is summable by Jackson's theorem $[2,238,eqn(II-20)]$. Thus, $\beta_{d,k,m}(a,q)$reduces to a finite product, and upon substituting it in (1.5) the left hand side of the resulting $a-RRT$identity will be a single-fold sum.

Definition: For $k\geq 1$, and $1\leq i\leq k$,

$$Q_{d,k,i}(a)=Q_{d,k,i}(a,q)=\frac{1}{(aq;q)_\infty}\sum_{n\geq 0}\frac{(-1)^na^{kn}q^{\left(dk+\frac{d}{2}\right)n^2+\left(k-i+\frac{1}{2}\right)dn}(1-a^iq^{(2n+1)di})(aq^d;q^d)_n}{(q^d;q^d)_n}$$

In [1], Andrew V. Sills has derived the following results with incorporation of the parameterized Bailey pairs and some q-difference equations as noted in [1].

Theorem 1.1: For $i = 1,2$ (*see [1, Theorem 3.6, p. 13]*)

$$F_{2,2,i}(a,q) = Q_{2,2,i}(a,q) \tag{1.7}$$

where,

$$F_{2,2,1}(a,q) = \sum_{n=0}^{\infty} \frac{a^n q^{\frac{3}{2}n^2+\frac{3}{2}n}}{(aq;q^2)_{n+1}(q;q)_n}$$

$$F_{2,2,2}(a,q) = \sum_{n=0}^{\infty} \frac{a^n q^{\frac{3}{2}n^2-\frac{1}{2}n}}{(aq;q^2)_n(q;q)_n}$$

Theorem 1.2: For $i = 1,2,3$ (*see [1, Theorem 3.9, p. 16]*)

$$F_{2,3,i}(a,q) = Q_{2,3,i}(a,q) \tag{1.8}$$

where,

$$F_{2,3,1}(a,q) = \sum_{n=0}^{\infty} \frac{a^n q^{n^2+2n}}{(aq;q^2)_{n+1}(q;q)_n}$$

$$F_{2,3,2}(a,q) = \sum_{n=0}^{\infty} \frac{a^n q^{n^2+n}}{(aq;q^2)_{n+1}(q;q)_n} \text{ and}, F_{2,3,3}(a,q) = \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(aq;q^2)_n(q;q)_n}$$

Theorem 1.3: For $i = 1,2,3,4$ (see [1, Theorem 3.12, p. 17])

$$F_{2,4,i}(a,q) = Q_{2,4,i}(a,q) \tag{1.9}$$

where,

$$F_{2,4,1}(a,q) = \sum_{n\geq 0}\sum_{r\geq 0} \frac{a^{n+r} q^{n^2+2n+2r^2+2r}}{(aq;q^2)_{n+1}(q;q)_{n-2r}(q^2;q^2)_r}$$

$$F_{2,4,2}(a,q) = \sum_{n\geq 0}\sum_{r\geq 0} \frac{a^{n+r} q^{n^2+2n+2r^2+2r}(1+aq^{2r+2})}{(aq;q^2)_{n+1}(q;q)_{n-2r}(q^2;q^2)_r}$$

$$F_{2,4,3}(a,q) = \sum_{n\geq 0}\sum_{r\geq 0} \frac{a^{n+r} q^{n^2+2r^2+2r}}{(aq;q^2)_n(q;q)_{n-2r}(q^2;q^2)_r}$$

$$F_{2,4,4}(a,q) = \sum_{n\geq 0}\sum_{r\geq 0} \frac{a^{n+r} q^{n^2+2r^2}}{(aq;q^2)_n(q;q)_{n-2r}(q^2;q^2)_r}$$

Theorem 1.3: For $i = 1,2,3$ (see *[1, Theorem 3.14, p. 14])*

$$F_{3,3,i}(a,q) = Q_{3,3,i}(a,q) \tag{1.10}$$

where,

$$F_{3,3,1}(a,q) = \sum_{n\geq 0}\sum_{r\geq 0} \frac{(-1)^r a^n q^{n^2+3n+3r(r-1)/2}(aq^3;q^3)_{n-r}}{(aq;q)_{2n+2}(q;q)_{n-2r}(q^3;q^3)_r}$$

$$F_{3,3,2}(a,q) = \sum_{n\geq 0}\sum_{r\geq 0} \frac{(-1)^r a^{n-1} q^{n^2+\frac{3r(r-3)}{2}}(a;q^3)_{n-r}(1+aq^{3r}-q^{3r})}{(aq;q)_{2n}(q;q)_{n-3r}(q^3;q^3)_r}$$

$$F_{3,3,3}(a,q) = \sum_{n\geq 0}\sum_{r\geq 0} \frac{(-1)^r a^n q^{n^2+3r(r-1)/2}(a;q^3)_{n-r}}{(aq;q)_{2n-1}(q;q)_{n-3r}(q^3;q^3)_r}$$

2. In this section, we derive some transformations related to the basic hyper geometric series by using (1.7)-(1.10).:

Setting $i = 1$ and $q = q^{1/2}, q^{3/2}$ successively in (1.7), it gives

$$\sum_{n=0}^{\infty} \frac{a^n q^{(3n^2+3n)/4}}{(aq^{1/2};q)_{n+1}(\boldsymbol{q}^{1/2};\boldsymbol{q}^{1/2})_n} = \frac{1}{(a\boldsymbol{q}^{1/2};\boldsymbol{q}^{1/2})_\infty} \sum_{n\geq 0} \frac{(-1)^n a^{2n} q^{\frac{5n^2}{2}+\frac{3n}{2}}(1-aq^{(2n+1)})(aq;q)_n}{(q;q)_n} \tag{2.1}$$

$$\sum_{n=0}^{\infty} \frac{a^n q^{(9n^2+9n)/4}}{(aq^{3/2};q^3)_{n+1}(q^{3/2};q^{3/2})_n} = \frac{1}{(aq^{3/2};q^{3/2})_\infty} \sum_{n\geq 0} \frac{(-1)^n a^{2n} q^{\frac{15n^2}{2}+\frac{9n}{2}}(1-aq^{6n+3})(aq^3;q^3)_n}{(q^3;q^3)_n} \tag{2.2}$$

For $i=2$ and $q=q^{\frac{1}{2}},q^{\frac{3}{2}}$ successively in (1.7), it yields

$$\sum_{n=0}^{\infty}\frac{a^n q^{\frac{3}{4}n^2-\frac{1}{4}n}}{(aq^{1/2};q)_{n+1}(\boldsymbol{q^{\frac{1}{2}}};\boldsymbol{q^{\frac{1}{2}}})_n}=\frac{1}{(a\boldsymbol{q^{\frac{1}{2}}};\boldsymbol{q^{\frac{1}{2}}})_\infty}\sum_{n\geq 0}\frac{(-1)^n a^{2n}q^{\frac{5n^2}{2}+\frac{n}{2}}(1-a^2q^{(4n+2)})(aq;q)_n}{(q;q)_n} \tag{2.3}$$

$$\sum_{n=0}^{\infty}\frac{a^n q^{(9n^2-3n}/4}{(aq^{3/2};q^3)_{n+1}(q^{3/2};q^{3/2})_n}=\frac{1}{(aq^{3/2};q^{3/2})_\infty}\sum_{n\geq 0}\frac{(-1)^n a^{2n}q^{\frac{15n^2}{2}+\frac{3n}{2}}(1-a^2q^{12n+6})(aq^3;q^3)_n}{(q^3;q^3)_n} \tag{2.4}$$

Setting $i=1$ and $q=q^{\frac{1}{2}},q^{\frac{3}{2}}$ successively in (1.8), it gives

$$\sum_{n=0}^{\infty}\frac{a^n q^{(n^2+2n)/2}}{(aq^{1/2};q)_{n+1}(q^{1/2};q^{1/2})_n}=\frac{1}{(aq^{1/2};q^{1/2})_\infty}\sum_{n\geq 0}\frac{(-1)^n a^{3n}q^{(7n^2+5n)/2}(1-aq^{(2n+1)})(aq;q)_n}{(q;q)_n} \tag{2.5}$$

$$\sum_{n=0}^{\infty}\frac{a^n q^{(3n^2+6n)/2}}{(aq^{3/2};q^3)_{n+1}(q^{3/2};q^{3/2})_n}=\frac{1}{(aq^{3/2};q^{3/2})_\infty}\sum_{n\geq 0}\frac{(-1)^n a^{3n}q^{(21n^2+15n)/2}(1-aq^{(6n+3)})(aq^3;q^3)_n}{(q^3;q^3)_n} \tag{2.6}$$

Setting $i=2$ and $q=q^{\frac{1}{2}},q^{\frac{3}{2}}$ successively in (1.8), it gives

$$\sum_{n=0}^{\infty}\frac{a^n q^{(n^2+n)/2}}{(aq^{1/2};q)_{n+1}(q^{1/2};q^{1/2})_n}=\frac{1}{(aq^{1/2};q^{1/2})_\infty}\sum_{n\geq 0}\frac{(-1)^n a^{3n}q^{(7n^2+3n)/2}(1-a^2q^{(4n+2)})(aq;q)_n}{(q;q)_n} \tag{2.7}$$

$$\sum_{n=0}^{\infty}\frac{a^n q^{(3n^2+3n)/2}}{(aq^{3/2};q^3)_{n+1}(q^{3/2};q^{3/2})_n}=\frac{1}{(aq^{3/2};q^{3/2})_\infty}\sum_{n\geq 0}\frac{(-1)^n a^{3n}q^{(21n^2+9n)/2}(1-a^2q^{(12n+6)})(aq^3;q^3)_n}{(q^3;q^3)_n} \tag{2.8}$$

Setting $i=3$ and $q=q^{\frac{1}{2}},q^{\frac{3}{2}}$ successively in (1.8), it gives

$$\sum_{n=0}^{\infty}\frac{a^n q^{\frac{n^2}{2}}}{(aq^{1/2};q)_n(q^{1/2};q^{1/2})_n}=\frac{1}{(aq^{1/2};q^{1/2})_\infty}\sum_{n\geq 0}\frac{(-1)^n a^{3n}q^{(7n^2+n)/2}(1-a^3q^{(6n+3)})(aq;q)_n}{(q;q)_n} \tag{2.9}$$

$$\sum_{n=0}^{\infty}\frac{a^n q^{\frac{3n^2}{2}}}{(aq^{3/2};q^3)_n(q^{3/2};q^{3/2})_n}=\frac{1}{(aq^{3/2};q^{3/2})_\infty}\sum_{n\geq 0}\frac{(-1)^n a^{3n}q^{(21n^2+3n)/2}(1-a^3q^{(18n+9)})(aq^3;q^3)_n}{(q^3;q^3)_n} \tag{2.10}$$

3. Main results:

Rogers-Ramanujan Type Identities Modulo 5:

Setting $a=1,q$ successively in the transformation (2.1) and then using (1.1), the following identities of Rogers-Ramanujan Type can be obtained,

$$\begin{aligned}(q^{1/2};q^{1/2})_\infty\sum_{n=0}^{\infty}\frac{q^{(3n^2+3n)/4}}{(q^{1/2};q)_{n+1}(q^{1/2};q^{1/2})_n}&=\sum_{n=0}^{\infty}(-1)^n q^{\frac{5n^2+3n}{2}}(1-q^{(2n+1)})\\&=\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{5n^2+3n}{2}}\\&=\prod_{n=0}^{\infty}\frac{1}{1-q^n},\text{ where } n\not\equiv 0,1,4 \pmod 5\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}(\boldsymbol{q^{\frac{1}{2}}};\boldsymbol{q^{\frac{1}{2}}})_\infty\sum_{n=0}^{\infty}\frac{q^{(3n^2+7n+4)/4}}{(q^{1/2};q)_{n+2}(\boldsymbol{q^{\frac{1}{2}}};\boldsymbol{q^{\frac{1}{2}}})_n}&=\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{5n^2-n}{2}}+q\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{5n^2+7n}{2}}\\&=\prod_{n=0}^{\infty}\frac{1}{1-q^n}+q\prod_{n=0}^{\infty}\frac{1}{1-q^n}\end{aligned}$$

(3.2)

where $n\not\equiv 0,2,3 \pmod 5$ and $n\not\equiv 0,1,4 \pmod 5$ respectively.

The transformation (2.3) for $a=1,q$ yields,

$$\begin{aligned}(q^{1/2};q^{1/2})_\infty\sum_{n=0}^{\infty}\frac{q^{(3n^2-n)/4}}{(q^{1/2};q)_{n+1}(q^{1/2};q^{1/2})_n}&=\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{5n^2+n}{2}}\\&=\prod_{n=0}^{\infty}\frac{1}{1-q^n},\text{ where } n\not\equiv 0,2,3 \pmod 5\end{aligned} \tag{3.3}$$

and

$$(q^{1/2};q^{1/2})_\infty \sum_{n=0}^{\infty} \frac{q^{(3n^2+3n)/4}}{(q^{1/2};q)_{n+2}(q^{1/2};q^{1/2})_n} = 1 + \prod_{n=0}^{\infty} \frac{1}{1-q^n}, \text{ where } n \not\equiv 0,1,4 \pmod 5 \quad (3.4)$$

Rogers-Ramanujan Type Identities Modulo 7:

The transformation (2.5) for $a = 1, q$ yields

$$(q^{1/2};q^{1/2})_\infty \sum_{n=0}^{\infty} \frac{q^{(n^2+2n)/2}}{(q^{1/2};q)_{n+1}(q^{1/2};q^{1/2})_n} = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{7n^2+5n}{2}}$$

$$= \prod_{n=0}^{\infty} \frac{1}{1-q^n}, \text{ where } n \not\equiv 0,1,6 \pmod 7 \quad (3.5)$$

and

$$(q^{1/2};q^{1/2})_\infty \sum_{n=0}^{\infty} \frac{q^{(n^2+4n+2)/2}}{(q^{1/2};q)_{n+2}(q^{1/2};q^{1/2})_n} = q^2 \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{7n^2+11n}{2}} + \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{7n^2+n}{2}}$$

$$= \prod_{n=0}^{\infty} \frac{1}{1-q^n} + q^2 \prod_{n=0}^{\infty} \frac{1}{1-q^n} \quad (3.6)$$

where $n \not\equiv 0,3,4 \pmod 7$ and $n \not\equiv 0,2,5 \pmod 7$ respectively.

The transformation (2.7) for $a = 1, q$ yields

$$(q^{1/2};q^{1/2})_\infty \sum_{n=0}^{\infty} \frac{q^{(n^2+n)/2}}{(q^{1/2};q)_{n+1}(q^{1/2};q^{1/2})_n} = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{7n^2+3n}{2}}$$

$$= \prod_{n=0}^{\infty} \frac{1}{1-q^n}, \text{ where } n \not\equiv 0,2, \pmod 7 \quad (3.7)$$

$$(q^{1/2};q^{1/2})_\infty \sum_{n=0}^{\infty} \frac{q^{(n^2+3n+2)/2}}{(q^{1/2};q)_{n+2}(q^{1/2};q^{1/2})_n} = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{7n^2+3n}{2}} + q \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{7n^2+9n}{2}}$$

$$= \prod_{n=0}^{\infty} \frac{1}{1-q^n} + q \prod_{n=0}^{\infty} \frac{1}{1-q^n} \quad (3.8)$$

where $n \not\equiv 0,3,4 \pmod 7$ and $n \not\equiv 0,1,6 \pmod 7$ respectively.

The transformation (2.9) for $a = 1, q$ yields

$$(q^{1/2};q^{1/2})_\infty \sum_{n=0}^{\infty} \frac{q^{\frac{n^2}{2}}}{(q^{1/2};q)_n(q^{1/2};q^{1/2})_n} = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{7n^2+n}{2}} = \prod_{n=0}^{\infty} \frac{1}{1-q^n}, n \not\equiv 0,3,4 \pmod 7 \quad (3.9)$$

$$(q^{\frac{1}{2}};q^{\frac{1}{2}})_\infty \sum_{n=0}^{\infty} \frac{q^{(n^2+2n)/2}}{(q^{\frac{1}{2}};q)_{n+1}(q^{\frac{1}{2}};q^{\frac{1}{2}})_n} = 1 + \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{7n^2+5n}{2}}, n \not\equiv 0,1,6 \pmod 7 \quad (3.10)$$

Rogers-Ramanujan Type Identities Modulo 15:

The transformation (2.2) for $a = 1, q^3$ yields,

$$(q^{3/2};q^{3/2})_\infty \sum_{n=0}^{\infty} \frac{q^{9(n^2+n)/4}}{(q^{3/2};q^3)_{n+1}(q^{3/2};q^{3/2})_n} = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{15n^2+9n}{2}}$$

$$= \prod_{n=0}^{\infty} \frac{1}{1-q^n}, \text{ where } n \not\equiv 0,3,12 \pmod{15} \quad (3.11)$$

and,

$$\rightarrow (q^{3/2};q^{3/2})_\infty \sum_{n=0}^{\infty} \frac{q^{(9n^2+21n+12)/4}}{(q^{3/2};q^3)_{n+2}(q^{3/2};q^{3/2})_n} = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{15n^2+3n}{2}} + q^3 \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{15n^2+21n}{2}}$$

$$= \prod_{n=0}^{\infty} \frac{1}{1-q^n} + q^3 \prod_{n=0}^{\infty} \frac{1}{1-q^n} \quad (3.12)$$

where $n \not\equiv 0,6,9 \pmod{15}$ and $n \not\equiv 0,3,12 \pmod{15}$ respectively.

The transformation (2.4) for $a = 1, q^3$ yields

$$(q^{3/2};q^{3/2})_\infty \sum_{n=0}^{\infty} \frac{q^{(9n^2-3n)/4}}{(q^{3/2};q^3)_{n+1}(q^{3/2};q^{3/2})_n} = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{15n^2+3n}{2}}$$

$$= \prod_{n=0}^{\infty} \frac{1}{1-q^n}, \text{ where } n \not\equiv 0,6,9 \pmod{15} \qquad (3.13)$$

and

$$(q^{3/2};q^{3/2})_\infty \sum_{n=0}^{\infty} \frac{q^{(9n^2+3n)/4}}{(q^{3/2};q^3)_{n+2}(q^{3/2};q^{3/2})_n} = 1+\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{15n^2+9n}{2}}$$

$$= 1+\prod_{n=0}^{\infty} \frac{1}{1-q^n}, \text{ where } n \not\equiv 0,3,12 \pmod{15} \qquad (3.14)$$

Rogers-Ramanujan Type Identities Modulo 21:

The transformation (2.6) for $a = 1, q^3$ yields

$$(q^{3/2};q^{3/2})_\infty \sum_{n=0}^{\infty} \frac{q^{(3n^2+6n)/2}}{(q^{3/2};q^3)_{n+1}(q^{3/2};q^{3/2})_n} = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{21n^2+15n}{2}}$$

$$= \prod_{n=0}^{\infty} \frac{1}{1-q^n}, \text{ where } n \not\equiv 0,3,18 \pmod{21} \qquad (3.15)$$

and

$$(q^{3/2};q^{3/2})_\infty \sum_{n=0}^{\infty} \frac{q^{(3n^2+12n+12)/2}}{(q^{3/2};q^3)_{n+2}(q^{3/2};q^{3/2})_n} = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{21n^2-3n}{2}} + q^6 \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{21n^2+33n}{2}} +$$

$$= \prod_{n=0}^{\infty} \frac{1}{1-q^n} + q^6 \prod_{n=0}^{\infty} \frac{1}{1-q^n} \qquad (3.16)$$

Where $n \not\equiv 0,9,12 \pmod{21}$ and $n \not\equiv 0,6,15 \pmod{21}$ respectively.

The transformation (2.8) for $a = 1, q^3$ yields

$$(q^{3/2};q^{3/2})_\infty \sum_{n=0}^{\infty} \frac{q^{(3n^2+3n)/2}}{(q^{3/2};q^3)_{n+1}(q^{3/2};q^{3/2})_n} = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{21n^2+9n}{2}}$$

$$= \prod_{n=0}^{\infty} \frac{1}{1-q^n}, \text{ where } n \not\equiv 0,6,15 \pmod{21} \qquad (3.17)$$

$$(q^{3/2};q^{3/2})_\infty \sum_{n=0}^{\infty} \frac{q^{(3n^2+9n+6)/2}}{(q^{3/2};q^3)_{n+2}(q^{3/2};q^{3/2})_n} = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{21n^2+9n}{2}} + q^3 \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{21n^2+27n}{2}}$$

$$= \prod_{n=0}^{\infty} \frac{1}{1-q^n} + q^3 \prod_{n=0}^{\infty} \frac{1}{1-q^n} \qquad (3.18)$$

where $n \not\equiv 0,3,18 \pmod{21}$ and $n \not\equiv 0,1,6 \pmod{21}$ respectively.

Finally, the transformation (2.10) for $a = 1, q^3$ yields

$$(q^{3/2};q^{3/2}) \sum_{n=0}^{\infty} \frac{q^{(3n^2+2n)/2}}{(q^{3/2};q^3)_n(q^{3/2};q^{3/2})_n} = \sum_{n=0}^{\infty}(-1)^n q^{(21n^2+3n)/2}(1-q^{(18n+9)})$$

$$= \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{21n^2+3n}{2}}$$

$$= \prod_{n=0}^{\infty} \frac{1}{1-q^n}, \textit{ where } n \not\equiv 0,9,12 \pmod{21} \qquad (3.19)$$

and

$$(q^{3/2};q^{3/2})_\infty \sum_{n=0}^{\infty} \frac{q^{(3n^2+6n)/2}}{(q^{3/2};q^3)_{n+1}(q^{3/2};q^{3/2})_n} = 1+\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{21n^2+15n}{2}}$$

$$= \prod_{n=0}^{\infty} \frac{1}{1-q^n}, \text{ where } n \not\equiv 0,3,18 \pmod{21} \qquad (3.20)$$

Conclusion:

Some other identities may be found after being replaced a by with some other indexes. Also there is a scope of obtaining more identities by incorporating some particular identities from the Slatter's famous list of 130 identities of Rogers-Ramanujan type.

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