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The Homotopy Perturbation Sumudu Transform Method for Solving the Generalized Time-Space Schrödinger Equation

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Abstract
Here has been presented and analyze the homotopy perturbation smudge transform method (HPSTM), which combines the homotopy perturbation method and the smudge transform method, and used this technique to solve the generalized time-space fractional Schrödinger equation an example to check the efficiency of the proposed method.

Keywords: caputo derivative, sumudu transform, homotopy perturbation method, Schrödinger equation

1. Introduction

While the mathematical models of derivatives with the integer order (linear and nonlinear) failed to address many scientific problems, this is why the emergence of fractional differentiation was more useful and important in several subjects, including mechanics, electricity, biology and economics [34, 39, 19]. The more general reason was to allow the required rate of change to be managed based on the requirements of a particular situation. Leibniz began studying fractional differentiation through observations of a differential of the order of 0.5 in the seventeenth century, and the scientist Louisville defined in a series of research papers (1832-1837) the fractional factor to open the way for the study of fractional differentiation (integration) of incorrect orders on the entire complex level

There are many methods that have emerged to generalize the idea of fractional differentiation, including Riemann-Louisville, Grünwald-Lietnikow, Caputo[35] and generalized functions, but they were not sufficient in the fields of physics, so Caputo presented an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations[6, 3].

Therefore, the numerical solutions of fractional partialdifferential equations (FPDE) have become the subject of interest and study in recent times, and accordingly, many fractional numerical methods have appeared to treat fractional differential equations (partial and ordinary), including classical solution techniques, including theFourier transform method [23], the Laplace transform method, and the green equation method, Mellon transform, orthogonalpolynomials [35, 13, 16, 27].

Recently, commonly used methods have appeared; which are introduced in this work, such as; Adomiandecomposition method (ADM)[28, 29, 30, 41], the homotopy perubation method (HPM) has prposed by He [13, 15], the modified homotopy perubationmethod (MHPM) [32], the differential transform method (DTM) [27], Variational Iteration Method (VIM) [5, 7], the homotopy analysis method (HAM) [2], the sumudu decomposition method [8], Fractional Difference Method (FDM) [29 , 28, 37, 17], and Power Series Method [34, 6, 29, 41, 30, 28, 17].

It also emerged from the homotopy pertubation method (HPM) of many methods that showed their effectiveness in fractional cases when combined with the variational iteration method (VIM) [32] to solve some nonlinear problems. The homotopy perturbation method (HPM) is also combined with the Laplace transform method [22], to obtain exact and approximate solutions for nonlinear equations. Likewise, the homotopy perturbation method (HPM) is combined with the double sumudu transform to have the homotopy perturbation double Sumudu transform method (HPDSM) to obtain the exact analytical and approximate solutions [37].
Also, the sumudu decomposition method (SDM) combines the sumudu transform and homotopy perturbation method [20]. Recently, a combination of the sumudu transformation method and homotopy perturbation methods to solve nonlinear problems is known as Homotopy perturbation sumudu transform method (HPSTM) [18, 40]. Many researchers use this method (HPSTM) to find the exact analytical and approximate solutions. Singh et al [41] have made use of studying the solution of linear (nonlinear) partial differential equations by using the homotopy perturbation sumudu, where the nonlinear part can be easily handled by He's polynomials [11].
In this studyapplied the homotopy perturbation sumudu transform method (HPSTM) to solve the generalized time-space fractional Schrödinger equation [17], with variable coefficients:

$$i\frac{\partial w^{\alpha}}{\partial t^{\alpha}}+a\frac{\partial w^{2\beta}}{\partial x^{2\beta}}+v(x)w+\gamma\left|w\right|^{2}w=0 \qquad (1)$$

Where $t>0,\ 0<\alpha\ ,\beta\leq 1$

With initial conditions $w\left(x,0\right)=w(x)$

Where is: $w\left(x,t\right)$ the unknown function.

$v\left(x\right)$: is the trapping potential $0<\alpha,\beta\leq 1$ are parameters describing the order of the fractional jumaries [17].

a,γ: are real contents respectively, so in section (2) some definitions of fractional calculus theory, in section (3) describe the homotopy partition sumudu method, In section (4) it contains the main results and an example to show the efficiency of using HPSTM, and in section (5) produced the conclusion.

2. Basic Definitions of Fractional Calculus

Definition 1: the Riemann-liouville fractional integral operator of order α >0, of a function $f(\mathrm{t})\in C_{u}$, $\mu\geq -1$, is defined as :

$$J^{a}f(\mathrm{t})=\frac{1}{\Gamma(\alpha)}\int_{0}^{t}\left(t-\tau\right)^{\alpha-1}f(\tau)\,\mathrm{d}\,\tau \qquad (2)$$

$$J^{0}f(t)=f(t).$$

Definition 2: The fractional derivative of $f(\mathrm{t})$ in the caputo sense is defined as [30, 32]

$$\mathrm{D}_{t}^{\alpha}f(\mathrm{t})=\mathrm{J}^{m-\alpha}D^{n}f(\mathrm{t})$$
$$=\frac{1}{\Gamma(m-\alpha)}\int_{0}^{t}(t-\tau)^{m-\alpha-1}f^{(\mathrm{m})}(\tau)\,\mathrm{d}\,\tau \qquad (3)$$

For $m-1<\alpha\leq m, m\in N, t>0$ and $\Gamma(\alpha)$ is the Gamma function.

Definition 3: the sumudu transform is defined over the set of functions :

$$A = \left\{ f(t) \mid \exists M, \tau_1, \tau_2 > 0, \left| f(t) \right| < Me^{\frac{|t|}{\tau_j}} \quad \text{if } t \in (-1)^j \times [0, \infty) \right\} \tag{4}$$

By:

$$\overline{f}(u) = S[f(t)] = \int_0^\infty f(ut) e^{-t} dt \quad , \ u \in (\tau_1, \tau_2) \tag{5}$$

Some special properties of sumudu transform are as follows:

$$S[1] = 1$$

$$S\left[\frac{t^m}{\Gamma(m+1)}\right] = u^m ; m > 0$$

Other properties of the sumudu transform can be found in [4]

Definition 4: the sumudu transform of the caputo fractional derivative is defined as follows [12]

$$S\left[D_t^\alpha f(x,t)\right] = u^{-\alpha} S\left[f(x,t)\right] - \sum_{k=0}^{m-1} u^{-\alpha+k} f^{(k)}(0+) \tag{6}$$

$, m-1 < \alpha \le m$

3. The homotopy perturbation sumudu transform method (HPSTM):

To illustrate the technique of the solution there is the Schrödinger equation

$$i \frac{\partial w^\alpha}{\partial t^\alpha} + a \frac{\partial w^{2\beta}}{\partial x^{2\beta}} + v(x) w + \gamma |w|^2 w = 0 \tag{7}$$

Where: $t>0,\ 0<\alpha,\ \beta\le 1$

With initial conditions w(x, 0) = w(x)

If write equation (1) so will be:

$$D_t^\alpha w(x,t) = i\left(aD_x^{2\beta} + v(x)\right) w(x,t) + i\gamma N w(x,t) \tag{8}$$

Where $D_t^\alpha w(x,t)$: is the caputo fractional derivative of the function w(x, t), v(x), is the source term N is the general nonlinear differential operator.

Applying the sumudu transform (S) on both sides of equation (2)

$$S\left[D_t^\alpha w(x,t)\right] = iS\left[\left(aD_x^{2\beta} + v(x)\right) w(x,t)\right] + i\gamma S\left[N w(x,t)\right] \tag{9}$$

Using the differentiation property of the sumudu transform and the initial conditions in equation (1) , so

$$S[w(x,t)] = w_0(x,t) + iu^\alpha S\left[\left(aD_x^{2\beta} + v(x)\right) w(x,t)\right] + i\gamma u^\alpha S\left[N w(x,t)\right] \tag{10}$$

Operating the sumudu inverse on both sides of equation (4)

$$w(x,t) = w_0(x,t) + iS^{-1} u^\alpha S\left[\left(aD_x^{2\beta} + v(x)\right) w(x,t)\right] + i\gamma S^{-1} u^\alpha S\left[N w(x,t)\right] \tag{11}$$

Where $w_0(x,t)$ represented the initial condition.

Now apply the homotopy perturbation method (HPM):

$$w(x,t)=\sum_{n=0}^{\infty}p^n w_n(x,t) \tag{12}$$

And the nonlinear term can be decomposed as

$$Nw(x,t)=\sum_{n=0}^{\infty}p^n A_n \tag{13}$$

For some Adomian's polynomials A_n [10]

$$A_n=\frac{1}{n!}\frac{d^n}{dp^n}\left[N\left(\sum_{i=0}^{\infty}p^i w_i\right)\right]_{p=0}\quad,\quad n=0,1,2,\ldots \tag{14}$$

Substitution Equations (14), (15) in the equation (13):

$$w_0(x,t)+\sum_{n=-1}^{\infty}p^{n+1}w_{n+1}(x,t)=w_0(x)+iS^{-1}u^{\alpha}S\left[\left(aD_x^{2\beta}+v(x)\right)\sum_{n=0}^{\infty}p^n w_n(x,t)+\gamma\sum_{n=0}^{\infty}p^n A_n\right] \tag{15}$$

Equating the terms with identical powers of P, so the series of equations as follows:

$$\begin{aligned}
p^0 &: w_0(x,t)=w_0(x)\\
p^1 &: w_1(x,t)=iS^{-1}u^{\alpha}S\left[\left(aD_x^{2\beta}w_0(x,t)+v(x)w_0(x,t)\right)+\gamma A_0\right]\\
p^2 &: w_2(x,t)=iS^{-1}u^{\alpha}S\left[\left(aD_x^{2\beta}w_1(x,t)+v(x)w_1(x,t)\right)+\gamma A_1\right]\\
&\vdots\\
p^n &: w_n(x,t)=iS^{-1}u^{\alpha}S\left[\left(aD_x^{2\beta}w_{n-1}(x,t)+v(x)w_{n-1}(x,t)\right)+\gamma A_{n-1}\right]
\end{aligned} \tag{16}$$

The approximate the analytical solution $w(x,t)$ by the truncated series:

$$w(x,t)=\lim_{N\to 0}\sum_{n=0}^{N}p^n w_n(x,t) \tag{17}$$

4. Application

In this section applyed the HPSTM that presented in section 3 for the following example:

Example:

Consider the time-fractional NLS equation:

$$i\frac{\partial w^{\alpha}}{\partial t^{\alpha}}+\frac{1}{2}\frac{\partial w^{2\beta}}{\partial x^{2\beta}}-w\cos^2(x)-|w|^2 w=0 \tag{18}$$

Where $t>0,\quad 0<\alpha,\beta\le 1$, with initial conditions $w(x,0)=\sin x$

Then the equation:

$$D_t^{\alpha}w(x,t)=i\left(\frac{1}{2}D_x^{2\beta}-\cos^2 x\right)w(x,t)-iNw(x,t)$$

So by the formula in (17), can be get:

$$w_0(x,t)+\sum_{n=-1}^{\infty}p^{n+1}w_{n+1}(x,t)=\sin x+iS^{-1}u^{\alpha}S\left[\sum_{n=0}^{\infty}\left(\frac{1}{2}D_x^{2\beta}-\cos^2 x\right)w(x,t)-A_n\right] \tag{19}$$

Equating the terms with identical powers of p,

$$
\begin{aligned}
&p^0 : w_0(x,t) = \sin x \\
&p^1 : w_1(x,t) = \frac{it^\alpha}{\Gamma(\alpha+1)}\left[\frac{1}{2}\sin(x+\beta\pi) - \sin x\right] \\
&p^2 : w_2(x,t) = \frac{(it^\alpha)^2}{\Gamma(2\alpha+1)}\left[\frac{1}{4}\sin(x+2\beta\pi) - \sin(x+\beta\pi) + \sin x\right] \\
&p^3 : w_3(x,t) = \frac{(it^\alpha)^3}{\Gamma(3\alpha+1)}\left[\frac{1}{8}\sin(x+3\beta\pi) - \frac{3}{4}\sin(x+2\beta\pi) + \frac{3}{2}\sin(x+\beta\pi) - \sin x\right] \\
&p^4 : w_4(x,t) = \frac{(it^\alpha)^4}{\Gamma(4\alpha+1)}\begin{bmatrix}\frac{1}{16}\sin(x+4\pi\beta) - \frac{1}{2}\sin(x+3\pi\beta) \\ +\frac{3}{2}\sin(x+2\pi\beta) - 2\sin(x+\pi\beta) + \sin(x)\end{bmatrix}, \qquad (20) \\
&p^5 : w_5(x,t) = \frac{(it^\alpha)^5}{\Gamma(5\alpha+1)}\begin{bmatrix}\frac{1}{32}\sin(x+5\pi\beta) - \frac{5}{16}\sin(x+4\pi\beta) + \frac{5}{4}\sin(x+3\pi\beta) \\ -\frac{5}{2}\sin(x+2\pi\beta) + \frac{5}{2}\sin(x+\pi\beta) - \sin(x)\end{bmatrix}, \\
&\vdots
\end{aligned}
$$

The solution of equation (20) given by:

$$
\begin{aligned}
&w(x,t) = \lim_{N\to 0}\sum_{n=0}^{N} p^n w_n(x,t) = \\
&\sin x + \frac{it^\alpha}{\Gamma(\alpha+1)}\left[\frac{1}{2}\sin(x+\beta\pi) - \sin x\right] \qquad (21) \\
&+\frac{(it^\alpha)^2}{\Gamma(2\alpha+1)}\left[\frac{1}{4}\sin(x+2\beta\pi) - \sin(x+\beta\pi) + \sin x\right] + \ldots
\end{aligned}
$$

This is agreement with the result in reference [25]

Fig 1: Comparison between the real part of w_4 and the exact solution.

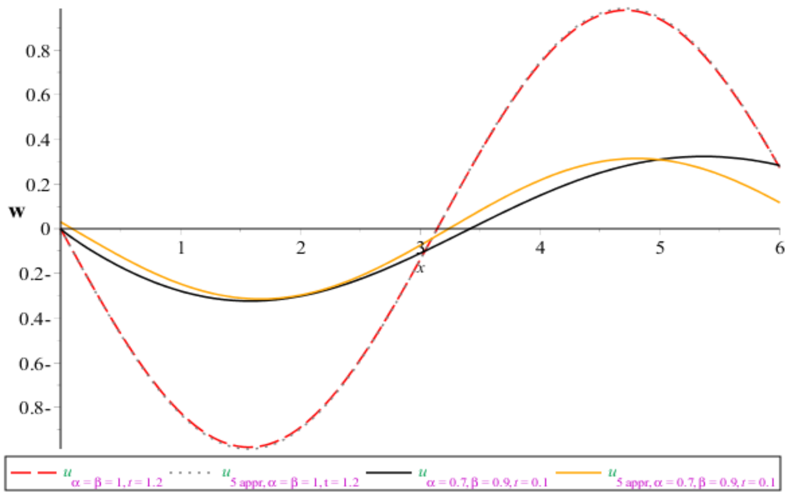

Fig 2: Comparison between the imaginary part of w_5 and the exact solution.

5. Conclusion:

In study presented a combination of the homotopy perturbation method (HPM), with the sumudu transformation method (STM). this combination build a strong method called the HPSTM, which is used in this work to solve the nonlinear fractional time-space Schrödinger equation, and the methed is so much easier to apply, and has been solve the equation effectively, easily with the approximations which convergent to exact solution.

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