Explicit Approximate Representations of Sequences as Suborbits in Banach Spaces: Universal and Function Space Approaches

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Abstract

In a recent extension of the work by Halperin et al., Grivaux demonstrated that any linearly independent sequence $\{f_k\}_{k=1}^{\infty}$ in a separable Banach space X can be expressed as a suborbit $\left\{T^{\alpha(k)}\varphi\right\}_{k=1}^{\infty}$ of some bounded operator $T: X \to X$. Typically, neither the operator T nor the powers $\alpha(k)$ are known explicitly. In this paper, we explore approximate representations $\{f_k\}_{k=1}^{\infty} \approx \left\{T^{\alpha(k)}\varphi\right\}_{k=1}^{\infty}$ for certain types of sequences $\{f_k\}_{k=1}^{\infty}$. Unlike previous results, we explicitly describe the operator T and the powers $\alpha(k)$, without requiring the sequences to be linearly independent. The notion of approximation is defined so that $\left\{T^{\alpha(k)}\varphi\right\}_{k=1}^{\infty}$ retains essential characteristics of $\{f_k\}_{k=1}^{\infty}$, such as in atomic decompositions and Banach frames.

We introduce two different approaches. The first is universal, applying to general Banach spaces. While the technical conditions are straightforward to verify in sequence spaces, they become more complex in function spaces. Therefore, we present a second approach, specifically designed for Banach function spaces. Several examples demonstrate that these results hold in arbitrary weighted ℓ^p-spaces and L^p-spaces.

Keywords: Approximate operator representations, Banach spaces, iterated systems, suborbits.

1 Introduction

A classical result by Halperin, Kitai, and Rosenthal [26] states that if $\{f_k\}_{k=1}^{\infty}$ is any linearly independent sequence in a separable Hilbert space $\mathcal{H}$, then there exists a bounded linear operator $T: \mathcal{H} \to \mathcal{H}$ and appropriate choices of powers $\alpha(k) \in \mathbb{N}_0$ such that $f_k = T^{\alpha(k)} f_1$, for all $k \in \mathbb{N}$.

This fundamental observation was later generalized to Banach spaces by Grivaux[23],using a different technique. Neither the operator T nor the appropriate powers $\alpha(k)$ are given in an easily accessible form in [23, 26]. Generalizing ideas presented in [12] in the setting of frames in Hilbert spaces, we will provide an alternative approach to the questions by Halpering et al. by considering approximate operator representations in a given Banach space X. In other words, we will give up the requirement that the operator T leads to an exact representation of the given sequence $\{f_k\}_{k=1}^{\infty}$. Instead, we will aim at a construction of a bounded operator T, a vector $\varphi \in X$, and appropriate powers $\alpha(k)$ such that the sequence $\left\{T^{\alpha(k)}\varphi\right\}_{k=1}^{\infty}$ approximates the given sequence $\{f_k\}_{k=1}^{\infty}$ in various senses to be specified below. We will show that in several cases we can specify as well the operator T, the vector φ, as the powers $\alpha(k)$.

The paper is organized as follows. In the rest of the introduction we set the stage by providing basic definitions and results concerning Banach sequence spaces and shift-operators on Banach spaces. In Sections 2.1-2.2 we provide general results for obtaining approximate representations of certain sequences, in the setting of general Banach spaces having a basis. The results are based on the assumption that the left/right-shift operators with respect to a certain basis - see (1.1) and (1.2) below - are bounded. This condition can easily be checked in several types of sequence spaces; in particular, given any weighted ℓ^p-space, we can specify a scaling of the canonical unit vectors that satisfy the condition. We can also verify the condition in certain Banach spaces of functions; however, the condition is difficult to handle in general function spaces. For this reason we provide an alternative approach, tailored to Banach function spaces, in Section 2.3. Here, the assumption of the left/right-shift operators being bounded is replaced by the condition that the translation operators act boundedly on the given function space, a condition that is trivially satisfied in, e.g., all weighted L^p-spaces with an m-moderate weight function. In Section 2.4 we show how to construct so-called ϵ-close approximations $\left\{T^{\alpha(k)}\varphi\right\}_{k=1}^{\infty}$ of the sequences $\{f_k\}_{k=1}^{\infty}$ discussed in Sections 2.1-2.3; this paves the way for the results in Section 2.5, where it is shown how to construct the sequence $\left\{T^{\alpha(k)}\varphi\right\}_{k=1}^{\infty}$ such that it keeps key features of the sequence $\{f_k\}_{k=1}^{\infty}$ in the setting of atomic decompositions.

In the entire paper X will denote a separable Banach space, and X_d will be a Banach space consisting of scalar-valued sequences indexed by $\mathbb{N}$. We will refer to such a space X_d as a Banach sequence space. We will need the following standard concept related to Banach sequence spaces.

Definition 1.1 Let X_d denote a Banach sequence space.
(i) X_d is said to be solid if whenever $\{c_k\}_{k=1}^{\infty} \in X_d$ and $\{b_k\}_{k=1}^{\infty}$ is any scalar-valued sequence such that $|b_k| \le |c_k|$ for all $k \in \mathbb{N}$, it follows that $\{b_k\}_{k=1}^{\infty} \in X_d$ and $\|\{b_k\}_{k=1}^{\infty}\| \le \|\{c_k\}_{k=1}^{\infty}\|$.
(ii) X_d is said to have an absolutely continuous norm if $\left\|\left\{c_k - c_k\chi_{I_n}(k)\right\}_{k=1}^{\infty}\right\| \to 0$ as $n \to \infty$ for any sequence $\{c_k\}_{k=1}^{\infty} \in X_d$ and any family of subsets $I_n \subset \mathbb{N}$ such that $I_1 \subset I_2 \dots \uparrow \mathbb{N}$.

The above concepts have parallel versions in Banach function space, defined by obvious modifications; we will apply these without further comments in the sequel.

1.1 Shift operators on Banach spaces

Recall that a sequence $\{e_k\}_{k=1}^{\infty}$ in X is a (Schauder) basis if every element $x \in X$ has a unique representation $x = \sum_{k=1}^{\infty} c_k e_k$ for some $c_k \in \mathbb{C}$. It was proved by Enflo [19] that not all the separable Banach spaces have a basis. Let X denote a Banach space having a basis $\{e_k\}_{k=1}^{\infty}$ and consider the left-shift and right-shift operators $L, R\colon \mathrm{span}\{e_k\}_{k=1}^{\infty} \to X$ given by

$$Le_1 = 0,\ Le_k = e_{k-1},\ k \ge 2 \tag{1.1}$$

respectively,

$$Re_k = e_{k+1},\ k \in \mathbb{N} \tag{1.2}$$

Throughout the paper we will need that the operators L, R extend to bounded linear operators on X. This condition in general depends not only on the space X but also on the choice of the basis $\{e_k\}_{k=1}^{\infty}$. We will show that the condition is satisfied if $\{e_k\}_{k=1}^{\infty}$ is a p-Riesz basis, a concept to be defined next.

Definition 1.2 Fix some $p \in [1, \infty)$. A sequence $\{e_k\}_{k=1}^{\infty} \subset X$ is called a pRiesz basis for X if $\operatorname{span}\{e_k\}_{k=1}^{\infty} = X$ and there exist constants $A, B > 0$ such that

$$A\left(\sum_{k=1}^{N} |c_k|^p\right)^{1/p} \leq \left\|\sum_{k=1}^{N} c_k e_k\right\| \leq B\left(\sum_{k=1}^{N} |c_k|^p\right)^{1/p} \tag{1.3}$$

for all finite scalar sequences $\{c_k\}_{k=1}^{N}, N \in \mathbb{N}$. The numbers A, B are called lower, resp. upper bounds.

Typically the lower condition in (1.3) is more involved to verify than the upper condition. A convenient criteria for the p-Riesz basis property, which avoids worrying about the lower bound, is stated next.

Lemma 1.3 Assume that $\{e_k\}_{k=1}^{\infty}$ is a basis for a reflexive Banach space X, with dual basis $\{e_k^*\}_{k=1}^{\infty}$. Assume that for some $p, q > 1$ with $p^{-1} + q^{-1} = 1$ there exists a constant $B > 0$ such that

$$\left(\sum_{k=1}^{\infty} |\langle f, e_k^*\rangle|^p\right)^{1/p} \leq B\|f\|, \ \forall f \in X$$

and

$$\left(\sum_{k=1}^{\infty} |\langle g, e_k\rangle|^q\right)^{1/q} \leq B\|g\|, \ \forall g \in X^*$$

Then $\{e_k\}_{k=1}^{\infty}$ is a p-Riesz basis for X and $\{e_k^*\}_{k=1}^{\infty}$ is a q-Riesz basis for X^*.
Proof. Let $N \in \mathbb{N}$ and $\{c_k\}_{k=1}^{N}$ be a finite scalar sequence. It follows by Hölder's inequality that

$$\begin{aligned}\left\|\sum_{k=1}^{N} c_k e_k\right\| &= \sup_{g \in X^*, \|g\|=1} \left|\left\langle\sum_{k=1}^{N} c_k e_k, g\right\rangle\right| \leq \sup_{g \in X^*, \|g\|=1} \sum_{k=1}^{N} |c_k||\langle e_k, g\rangle| \\ &\leq \sup_{g \in X^*, \|g\|=1} \left(\sum_{k=1}^{N} |c_k|^p\right)^{1/p} \left(\sum_{k=1}^{N} |\langle g, e_k\rangle|^q\right)^{1/q} \\ &\leq B\left(\sum_{k=1}^{N} |c_k|^p\right)^{1/p}.\end{aligned}$$

Also letting $f := \sum_{k=1}^{N} c_k e_k$, we have that $c_k = \langle f, e_k^*\rangle$ for $k = 1, \ldots, n$. Therefore

$$\left(\sum_{k=1}^{N} |c_k|^p\right)^{1/p} = \left(\sum_{k=1}^{N} |\langle f, e_k^*\rangle|^p\right)^{1/p} \leq B\|f\| = B\left\|\sum_{k=1}^{N} c_k e_k\right\|.$$

This proves that $\{e_k\}_{k=1}^{\infty}$ is a p-Riesz basis for X with bounds B^{-1} and B. The proof that $\{e_k^*\}_{k=1}^{\infty}$ is a q-Riesz basis for X^* is similar.

We now prove that if $\{e_k\}_{k=1}^{\infty}$ is a p-Riesz basis for some $p \in [1, \infty)$, then indeed the left/right-shift operators with respect to $\{e_k\}_{k=1}^{\infty}$ are bounded:

Proposition 1.4 If $\{e_k\}_{k=1}^{\infty}$ is a p-Riesz basis for X with bounds A, B for some $p \in [1, \infty)$, then the operators L, R in (1.1) and (1.2) extend to bounded linear operators on X, and

$$\|L\| \leq \frac{B}{A} \text{ and } \frac{A}{B} \leq \|R\| \leq \frac{B}{A}$$

Proof. Given any finite sequence $\{c_k\}_{k=1}^{N}$,

$$\begin{aligned}\left\|L \sum_{k=1}^{N} c_k e_k\right\| = \left\|\sum_{k=2}^{N} c_k e_{k-1}\right\| = \left\|\sum_{k=1}^{N-1} c_{k+1} e_k\right\| \\ \leq B\left(\sum_{k=1}^{N-1} |c_{k+1}|^p\right)^{1/p} \leq B\left(\sum_{k=1}^{N} |c_k|^p\right)^{1/p} \\ \leq \frac{B}{A}\left\|\sum_{k=1}^{N} c_k e_k\right\|\end{aligned}$$

Thus L extends to a bounded operator on X, with the claimed estimate of the norm. The proof for the boundedness of the right-shift operator and the upper estimate on its norm is similar. The lower bound on the norm of the operator R follows from L being a left-inverse of R, i.e., $\|f\| = \|LRf\| \leq \|L\|\|Rf\| \leq BA^{-1}\|Rf\|$ for all $f \in X$.

As an application of Proposition 1.4 we will now consider weighted ℓ^p spaces. Fixing any $p \in [1, \infty)$ and considering a sequence of positive scalars $\{w_k\}_{k=1}^{\infty}$, define the space ℓ_w^p by

$$\ell_w^p := \left\{\{c_k\}_{k=1}^{\infty} \mid c_k \in \mathbb{C} \text{ and } \sum_{k=1}^{\infty} |c_k|^p w_k < \infty\right\}.$$

Obviously, ℓ_w^p is a Banach space with respect to the norm

$$\|\{c_k\}_{k=1}^{\infty}\|_{p,w} := \left(\sum_{k=1}^{\infty} |c_k|^p w_k\right)^{1/p}$$

It is also clear that the canonical unit vectors $\{\delta_k\}_{k=1}^{\infty}$ form a basis for ℓ_w^p. The following result shows that in any weighted ℓ^p-space, we can specify a certain scaling of the canonical unit vectors that makes the left/right-shift operators bounded.

Corollary 1.5 Fix any $p \in [1, \infty)$, consider a sequence of positive scalars $\{w_k\}_{k=1}^{\infty}$, and let $\{\delta_k\}_{k=1}^{\infty}$ denote the canonical unit basis for ℓ_w^p. Let $e_k := \omega_k^{-1/p} \delta_k$. Then $\{e_k\}_{k=1}^{\infty}$ is a p-Riesz basis for ℓ_w^p with bounds $A = B = 1$. In particular, the left/right-shift operators L, R with respect to the basis $\{e_k\}_{k=1}^{\infty}$ are bounded, and $\|L\| = \|R\| = 1$.

Proof. A simple calculation shows that for any $N \in \mathbb{N}$ and any finite scalar sequence $\{c_k\}_{k=1}^{N}$,

$$\left\|\sum_{k=1}^{N} c_k e_k\right\|_{p,\omega}^{p} = \sum_{k=1}^{N} |c_k|^p$$

Now by Proposition 1.4 the stated results for the right-shift operator R follows immediately. It also follows that the left-shift operator L is bounded and that $\|L\| \leq 1$. Since $\|Le_2\|_{p,w} = \|e_1\|_{p,w} = \|e_2\|_{p,w}$, we finally conclude that $\|L\| = 1$, as claimed.

While the scaling of the basis in $\{\delta_k\}_{k=1}^{\infty}$ in Corollary 1.5 indeed makes the operators L and R bounded, the scaling might affect other conditions that are put on the basis, see., e.g., the condition (ii) in the forthcoming Theorem 2.2. For the case that it is most convenient to work with the canonical unit vector basis, we now characterize the weighted ℓ^p-spaces for which the left/right-shift operators are bounded with respect to $\{\delta_k\}_{k=1}^{\infty}$. We leave the proof to the reader.

Lemma 1.6 Fix any $p \in [1, \infty)$, consider a sequence of positive scalars $\{w_k\}_{k=1}^{\infty}$, and let $\{\delta_k\}_{k=1}^{\infty}$ denote the standard basis for ℓ_w^p. Then the following holds true:
(i) The left-shift operator $L\delta_1 = 0, L\delta_k = \delta_{k-1} \cdot k \geq 2$, extends to a bounded operator on ℓ_w^p if and only if $\sup_{k\geq 2} \frac{w_{k-1}}{w_k} < \infty$; in the affirmative case,

$$\|L\| = \sup_{k\geq 2} \left(\frac{w_{k-1}}{w_k}\right)^{1/p}$$

(ii) The right-shift operator $R\delta_k = \delta_{k+1}$ extends to a bounded operator on ℓ_w^p if and only if $\sup_{k\geq 2} \frac{w_k}{w_{k-1}} < \infty$; in the affirmative case,

$$\|R\| = \sup_{k\geq 2} \left(\frac{w_k}{w_{k-1}}\right)^{1/p}$$

Several constructions of p-Riesz bases are available in the literature. Let us mention an example in the setting of shift-invariant subspaces of $L^p(\mathbb{R})$:

Example 1.7 It is well-known [6] how to construct a function $\varphi \in L^2(\mathbb{R})$ such that the set of integer-translates $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ form a Riesz basis for the Hilbert space $S_2 := \overline{\operatorname{span}}\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$. Requiring furthermore that φ belongs to the Wiener space, i.e., that $\sum_{k\in\mathbb{Z}} \left\|\varphi\chi_{[k,k+1)}\right\|_\infty < \infty$, it is proved in [4] that for any $p \in [1, \infty)$ the family $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ is a p-Riesz basis for the subspace S_p of $L^p(\mathbb{R})$ given by

$$S_p := \left\{\sum_{k\in\mathbb{Z}} c_k\varphi(\cdot - k) \mid \{c_k\}_{k\in\mathbb{Z}} \in \ell^p(\mathbb{Z})\right\}$$

Similar results are known in the setting of modulation spaces, introduced by Feichtinger in [20].

Let us finally mention that the definition of p-Riesz bases can be generalized in an obvious way to the so-called X_d-Riesz bases [17]; here, the sequence space ℓ^p is simply replaced by a general Banach sequence space X_d. Rather than going for the highest level of abstraction we have decided to state the results in the setting of p-Riesz bases, because this setting allows us to be very explicit and hereby facilitates concrete applications.

2 Approximate operator representations

The goal of this section is to derive approximate representations of sequences $\{f_k\}_{k=1}^{\infty}$ in a separable Banach space X of the form $\left\{T^{\alpha(k)}\varphi\right\}_{k=1}^{\infty}$ for suitable choices of a vector $\varphi \in X$, a bounded operator $T: X \to X$, and the powers $\alpha(k), k \in \mathbb{N}$. We begin with the case where $\{f_k\}_{k=1}^{\infty}$ consists of "finite sequences" in Section 2.1. The results are generalized to "sufficiently fast decaying sequences" in Section 2.2. An alternative approach, tailored to the setting of Banach function spaces, is presented in Section 2.3. The purpose of Sections $2.4 - 2.5$ is to apply the results in Sections 2.1-2.3 to the setting of atomic decompositions and Banach frames: we show how to design the approximations such that the sequence $\left\{T^{\alpha(k)}\varphi\right\}_{k=1}^{\infty}$ keeps essential features of the given sequence $\{f_k\}_{k=1}^{\infty}$.

In order to facilitate reading of the next sections, we mention that the theoretical results in Sections 2.1-2.3 have a common structure: fixing arbitrary positive scalars $\{\epsilon_k\}_{k=1}^{\infty} \in \ell^1$ that are chosen according to the desired level of approximation, they show how we for a given sequence $\{f_k\}_{k=1}^{\infty} \subset X$ can choose a vector $\varphi \in X$, a bounded operator $T: X \to X$, and corresponding powers $\alpha(k), k \in \mathbb{N}$, such that

$$\left\| f_k - T^{\alpha(k)}\varphi \right\| \le \sum_{j=k+1}^{\infty} \epsilon_j \tag{2.1}$$

Since the positive scalars $\{\epsilon_k\}_{k=1}^{\infty} \in \ell^1$ are arbitrary, this implies that we for any given sequence $\{\mathcal{E}_k\}_{k=1}^{\infty}$ of positive scalars can obtain that $\left\| f_k - T^{\alpha(k)}\varphi \right\| \le \mathcal{E}_k$. Indeed, assuming without loss of generality that the sequence $\{\mathcal{E}_k\}_{k=1}^{\infty}$ is decreasing and that $\mathcal{E}_k \to 0$ as $k \to \infty$, we obtain this inequality by applying (2.1) to any sequence $\{\epsilon_k\}_{k=1}^{\infty} \in \ell^1$ such that $\epsilon_k \le \mathcal{E}_k - \mathcal{E}_{k+1}$ for all $k \in \mathbb{N}$. The consequences of an equality of the form (2.1) are considered in Sections 2.4-2.5.

2.1 Finite sequences in Banach spaces

Let X denote a separable Banach space having a basis $\{e_k\}_{k=1}^{\infty}$. As standing assumption we need that the operators L, R defined in (1.1) extend to bounded linear operators on X. Choose $\lambda > \|R\|$ and consider the bounded operators $T, S: X \to X$ given by

$$T = \lambda L,\ S = \lambda^{-1} R \tag{2.2}$$

Note that by the choice of λ we have $\|S\| < 1$, a condition that will be crucial in Theorem 2.1. We will first consider natural generalizations of finite sequences to the general Banach space X. Indeed, we will assume that each vector f_k is a finite linear combination of vectors from the basis $\{e_k\}_{k=1}^{\infty}$. Our approach and proof are inspired by a classical result by Rolewicz [28] in the setting of hypercyclic operators.

Theorem 2.1 In the above setup, consider a sequence $\{f_k\}_{k=1}^{\infty} \subset X$ and assume that for every $k \in \mathbb{N}$, there exists an integer $N(k)$ such that $f_k \in \operatorname{span}\{e_j\}_{j=1}^{N(k)}$. Fixing any sequence $\{\epsilon_k\}_{k=1}^{\infty} \in \ell^1$ consisting of positive numbers, choose a sequence $\{\alpha(k)\}_{k\in\mathbb{N}}$ of nonnegative integers such that

$$\alpha(k) - \alpha(k-1) \ge \max\left\{\frac{\ln\,\epsilon_k - \ln\|f_k\|}{\ln\,\|S\|}, N(k-1), N(k)\right\},\ k \ge 2 (2.3)$$

and let

$$\varphi := \sum_{j=1}^{\infty} S^{\alpha(j)} f_j \tag{2.4}$$

Then

$$\left\| f_k - T^{\alpha(k)}\varphi \right\| \le \sum_{j=k+1}^{\infty} \epsilon_j \tag{2.5}$$

Proof. For convenience of the proof, let

$$r_k := \max\left\{\frac{\ln\,\epsilon_k - \ln\|f_k\|}{\ln\,\|S\|}, N(k)\right\},\ k \in \mathbb{N}. \tag{2.6}$$

We first show that the vector φ in (2.4) is well-defined. To that end, let $\varphi_n := \sum_{j=1}^{n} S^{\alpha(j)} f_j, n \in \mathbb{N}$. Then, for any

$m, n \in \mathbb{N}$ with $m \leq n$, we have

$$\|\varphi_n - \varphi_m\| = \left\| \sum_{j=m}^{n} S^{\alpha(j)} f_j \right\| \leq \sum_{j=m}^{n} \|S\|^{\alpha(j)} \|f_j\|.$$

It follows from (2.6) that $\|S\|^{r_j}\|f_j\| \leq \epsilon_j$ for all $j \in \mathbb{N}$. Since $\|S\| < 1$ and $\alpha(k) > r_k$ by (2.3) and the definition of r_k, it follows that

$$\|\varphi_n - \varphi_m\| \leq \sum_{j=m}^{n} \|S\|^{r_j}\|f_j\| \leq \sum_{j=m}^{n} \epsilon_j \to 0 \ \text{ as } m, n \to \infty$$

Thus $\{\varphi_n\}_{n=1}^{\infty}$ is a Cauchy sequence and hence φ is well-defined. We now prove that (2.5) holds. Fix $k \in \mathbb{N}$ and consider $j \in \{1, \ldots, k-1\}$. The inequality (2.3) implies that

$$\alpha(k) - \alpha(j) > \alpha(j+1) - \alpha(j) \geq N(j)$$

Therefore for any $j < k$, we have $T^{\alpha(k)-\alpha(j)} f_j = 0$. Thus we can write

$$\begin{aligned}
\left\|f_k - T^{\alpha(k)}\varphi\right\| &= \left\| f_k - T^{\alpha(k)} \sum_{j=1}^{\infty} S^{\alpha(j)} f_j \right\| \\
&= \left\| \sum_{j=1}^{k-1} T^{\alpha(k)-\alpha(j)} f_j + \sum_{j=k+1}^{\infty} S^{\alpha(j)-\alpha(k)} f_j \right\| \\
&= \left\| \sum_{j=k+1}^{\infty} S^{\alpha(j)-\alpha(k)} f_j \right\| \\
&\leq \sum_{j=k+1}^{\infty} \|S\|^{\alpha(j)-\alpha(k)} \|f_j\|
\end{aligned}$$

Now for $j > k$, (2.3) implies

$$\alpha(j) - \alpha(k) \geq \alpha(j) - \alpha(j-1) \geq r_j.$$

Therefore

$$\left\|f_k - T^{\alpha(k)}\varphi\right\| \leq \sum_{j=k+1}^{\infty} \|S\|^{r_j}\|f_j\| \leq \sum_{j=k+1}^{\infty} \epsilon_j$$

as claimed.

The key condition in Theorem 2.1 is that the left/right-shift operators with respect to a certain basis are bounded. Recall that Corollary 1.5 shows how to fulfill this condition in any weighted ℓ^p-space, $1 \leq p < \infty$. It is typically significantly more complicated to verify boundedness of the shift operators on Banach function spaces than on Banach sequence spaces. For this reason we will formulate an alternative result in Section 2.3, tailored to the setting of Banach function spaces.

2.2 Localized sequences in Banach spaces

Next we will prove that Theorem 2.1 can be generalized to certain infinite sequences, provided their coordinates decay "sufficiently fast". To motivate the exact formulation of the condition, assume for a moment that $\{f_k\}_{k=1}^{\infty}$ is a sequence in ℓ^p for some $p \in (1, \infty)$. The canonical delta-sequence $\{e_k\}_{k=1}^{\infty}$ is an unconditional basis for ℓ^p and its dual space $(\ell^p)^*$, and the jth coordinate in the vector f_k is precisely $\langle f_k, e_j \rangle$. A natural way of defining "fast decay" of the coordinates of f_k is to require that there exist constants $C, \beta > 0$ such that

$$\left|\langle f_k, e_j \rangle\right| \leq Ce^{-\beta|j-k|},\ \forall j,k \in \mathbb{N} \tag{2.7}$$

We will use exactly this idea, but formulated for a general basis for the Banach space X.

Theorem 2.2 Let X denote a Banach space with basis $\{e_k\}_{k=1}^{\infty}$ and associated dual basis $\{e_k^*\}_{k=1}^{\infty}$, and let X_d be a solid Banach sequence space with an absolutely continuous norm, which contains the canonical unit vectors $\{\delta_k\}_{k=1}^{\infty}$. Let $\{f_k\}_{k=1}^{\infty} \subset X$ and assume the followings:
(i) The left/right-shift operators L, R with respect to the given basis $\{e_k\}_{k=1}^{\infty}$ are bounded on X. Choose any $\lambda > \|R\|$.
(ii) There exists a constant $B > 0$ such that

$$\left\|\sum_{k=1}^{\infty} c_k e_k\right\| \leq B\|\{c_k\}_{k=1}^{\infty}\|_{X_d}$$

for all finite sequences $\{c_k\}_{k=1}^{\infty}$.
(iii) There exist constants $C > 0$ and $\beta > \ln\,\lambda$ such that $\left\{e^{-\beta j}\right\}_{j=1}^{\infty} \in X_d$ and

$$\left|\langle f_k, e_j^* \rangle\right| \leq Ce^{-\beta|j-k|}\ \forall j,k \in \mathbb{N} \tag{2.8}$$

Finally, fixing a sequence $\{\epsilon_k\}_{k=1}^{\infty} \in \ell^1$ of positive scalars, choose a strictly increasing sequence of nonnegative integers $\{\alpha(k)\}_{k=1}^{\infty}$ such that for all $k \in \mathbb{N}$,

$$\|S\|^{\alpha(k)-\alpha(k-1)}\|f_k\|_X \leq \epsilon_k/2 \tag{2.9}$$

Then $\varphi := \sum_{k=1}^{\infty} S^{\alpha(k)} f_k$ is well-defined. Moreover, by choosing $\alpha(1) = 0$ and $\{\alpha(k)\}_{k=2}^{\infty}$ recursively such that

$$\alpha(k) \geq \alpha(k-1) + k - 2,\ \forall k \geq 2 \tag{2.10}$$

and

$$\alpha(k) > \frac{\ln\left(\sum_{j=k+1}^{\infty}\epsilon_j\right) - \ln\left(\left\|\{e^{-\beta j}\}_{j=1}^{\infty}\right\|_{X_d}\right) - \ln\left(\sum_{n=0}^{k-1}(\lambda e^{-\beta})^{-\alpha(n)}e^{\beta n}\right) - \ln(2BC)}{\ln(\lambda) - \beta},$$

then

$$\left\|f_k - T^{\alpha(k)}\varphi\right\|_X \leq \sum_{n=k+1}^{\infty} \epsilon_n,\ \forall k \in \mathbb{N}$$

Proof. First note that the infinite sum $\sum_{k=1}^{\infty} S^{\alpha(k)} f_k$ is absolutely convergent; indeed, by (2.9),

$$\sum_{k=1}^{\infty}\left\|S^{\alpha(k)} f_k\right\|_X \leq \sum_{k=1}^{\infty}\|S\|^{\alpha(k)}\|f_k\|_X \leq 1/2\sum_{k=1}^{\infty}\epsilon_k < \infty$$

Thus $\varphi = \sum_k S^{\alpha(k)} f_k$ is well-defined. Let $k \in \mathbb{N}$, then

$$\left\|f_k - T^{\alpha(k)}\varphi\right\|_X \le \sum_{n=1}^{k-1} \left\|T^{\alpha(k)}S^{\alpha(n)}f_n\right\|_X + \left\|\sum_{n=k+1}^{\infty} T^{\alpha(k)}S^{\alpha(n)}f_n\right\|_X \tag{2.12}$$

We study the two terms at the right-hand side of the inequality separately. First,

$$\begin{aligned}\left\|\sum_{n=k+1}^{\infty} T^{\alpha(k)}S^{\alpha(n)}f_n\right\|_X &= \left\|\sum_{n=k+1}^{\infty} S^{\alpha(n)-\alpha(k)}f_n\right\|_X \le \sum_{n=k+1}^{\infty} \left\|S^{\alpha(n)-\alpha(k)}f_n\right\|_X \\ &\le \sum_{n=k+1}^{\infty} \|S\|^{\alpha(n)-\alpha(k)}\|f_n\|_X \\ &\le \sum_{n=k+1}^{\infty} \|S\|^{\alpha(n)-\alpha(n-1)}\|f_n\|_X\end{aligned}$$

Using (2.9), we get

$$\left\|\sum_{n=k+1}^{\infty} T^{\alpha(k)}S^{\alpha(n)}f_n\right\|_X \le 1/2 \sum_{n=k+1}^{\infty} \epsilon_n \tag{2.13}$$

Now, for $n = 1, \dots, k-1$,

$$\begin{aligned}\left\|T^{\alpha(k)}S^{\alpha(n)}f_n\right\|_X &= \left\|T^{\alpha(k)}S^{\alpha(n)}\sum_{j=1}^{\infty} \langle f_n, e_j^*\rangle e_j\right\| \\ &= \left\|\lambda^{\alpha(k)-\alpha(n)} \sum_{j=\alpha(k)-\alpha(n)+1}^{\infty} \langle f_n, e_j^*\rangle e_{j-\alpha(k)+\alpha(n)}\right\| \\ &= \left\|\lambda^{\alpha(k)-\alpha(n)} \sum_{j=1}^{\infty} \langle f_n, e_{j+\alpha(k)-\alpha(n)}^*\rangle e_j\right\|.\end{aligned}$$

Using the condition (iii), we have $\left|\langle f_n, e_{j+\alpha(k)-\alpha(n)}^*\rangle\right| \le Ce^{-\beta|j+\alpha(k)-\alpha(n)-n|}$; since X_d is a solid Banach sequence space it follows that $\left\{\langle f_n, e_{j+\alpha(k)-\alpha(n)}^*\rangle\right\}_{k=1}^{\infty} \in X_d$ and that

$$\left\|\left\{\langle f_n, e_{j+\alpha(k)-\alpha(n)}^*\rangle\right\}_{j=1}^{\infty}\right\|_{X_d} \le C \left\|\left\{e^{-\beta|j+\alpha(k)-\alpha(n)-n|}\right\}_{j=1}^{\infty}\right\|_{X_d}$$

A standard argument (using that X_d is assumed to have an absolutely continuous norm) shows that the condition (ii) actually implies that the stated inequality holds for all $\{c_k\}_{k=1}^{\infty} \in X_d$; thus we arrive at

$$\left\|T^{\alpha(k)}S^{\alpha(n)}f_n\right\|_X \le BC\lambda^{\alpha(k)-\alpha(n)} \left\|\left\{e^{-\beta|j+\alpha(k)-\alpha(n)-n|}\right\}_{j=1}^{\infty}\right\|_{X_d}$$

Condition (2.10) implies that $j + \alpha(k) - \alpha(n) - n \ge 0$. Thus

$$\begin{aligned}\left\|T^{\alpha(k)}S^{\alpha(n)}f_n\right\|_X &\le BC\lambda^{\alpha(k)-\alpha(n)}\left\|\left\{e^{-\beta(j+\alpha(k)-\alpha(n)-n)}\right\}_{j=1}^{\infty}\right\|_{X_d}\\ &= BC\lambda^{\alpha(k)-\alpha(n)}\left\|e^{-\beta(\alpha(k)-\alpha(n)-n)}\left\{e^{-\beta j}\right\}_{j=1}^{\infty}\right\|_{X_d}\\ &\le BC\left(\lambda e^{-\beta}\right)^{\alpha(k)-\alpha(n)}e^{\beta n}\left\|\left\{e^{-\beta j}\right\}_{j=1}^{\infty}\right\|_{X_d}\end{aligned}$$

If $\{\alpha(k)\}_{k=1}^{\infty}$ satisfies the growth condition specified in (2.11), then we conclude

$$\sum_{n=1}^{k-1}\left\|T^{\alpha(k)}S^{\alpha(n)}f_n\right\|_X \le 1/2\sum_{j=k+1}^{\infty}\epsilon_j \tag{2.14}$$

The result now follows from (2.12), (2.13) and (2.14).

Example 2.3 Consider the Banach space $X=\ell_w^p$. As we proved in Corollary 1.5, the right/left-shift operators L,R defined with respect to the basis $\{e_k\}_{k=1}^{\infty}:=\left\{w_k^{-1/p}\delta_k\right\}_{k=1}^{\infty}$ are bounded and $\|L\|=\|R\|=1$. Let $X_d=\ell^p$. Clearly ℓ^p is a solid Banach space with absolutely continuous norm and it contains the canonical basis $\{\delta_k\}_{k=1}^{\infty}$. Moreover, for every finite sequence $\{c_k\}$,

$$\left\|\sum c_k e_k\right\|_X=\left\|\left\{c_k w_k^{-1/p}\right\}\right\|_X=\left(\sum |c_k|^p\right)^{1/p}=\|\{c_k\}\|_{X_d}$$

Therefore all the conditions in Theorem 2.2 are satisfied. The dual basis of $\{e_k\}_{k=1}^{\infty}$ is given by $e_k^*=w_k^{1/p}\delta_k, k\in\mathbb{N}$. Given any sequence $\{f_k\}_{k=1}^{\infty}\subset X$, as in Theorem 2.2, write $f_k=\left\{(f_k)_j\right\}_{j=1}^{\infty}$. Then

$$\left|\langle f_k,e_j^*\rangle\right|=\left|(f_k)_j\right|w_j^{1/p}, j,k\in\mathbb{N}$$

This shows that in the setting of $X=\ell_w^p, X_d=\ell^p$, Theorem 2.2 applies to all sequences $\{f_k\}_{k=1}^{\infty}$ such that for some $C,\beta>0$,

$$\left|(f_k)_j\right|\le Cw_j^{-1/p}e^{-\beta|j-k|},\ \forall j,k\in\mathbb{N}$$

2.3 Banach function spaces

The results in Sections 2.1-2.2 deal with general Banach spaces, having a basis with respect to which the shift operators L and R are bounded. This condition is often considerably more complicated to verify in Banach function spaces than in Banach sequence spaces. For this reason we will now consider the special case of Banach function spaces, but without any condition of knowledge of a basis such that the corresponding left/right-shift operators are bounded. In the entire section we let X denote a Banach space of functions $f:\mathbb{R}\to\mathbb{C}$. For $a\in\mathbb{R}$, consider the translation operator T_a acting on functions $f:\mathbb{R}\to\mathbb{C}$ by $T_af(x):=f(x-a)$. We say that X is translation invariant if the translation operators T_1 and T_{-1} map functions in X into X; in this case, if T_1 is bounded, it follows from the open mapping theorem that also T_{-1} is bounded. Assuming that X is a solid Banach space and the translation operator T_1 is bounded, consider now for any $\lambda>\|T_1\|$ the weighted translation operators $S,T:X\to X$,

$$Tf=\lambda(T_{-1}f)\chi_{[0,\infty)},\ Sf=\lambda^{-1}T_1f,\ f\in X \tag{2.15}$$

Note that $\|S\|<1$ and that for $k\in\mathbb{N}$, we have $T^kf=\lambda^k(T_{-k}f)\chi_{[0,\infty)}, f\in X$.
Before stating the main result, Theorem 2.4, let us comment on one of the conditions in the statement of the result and various ways of circumventing it. We will consider a sequence $\{f_k\}_{k=1}^{\infty}$ of function in X that are supported in $[0,\infty)$. First, the choice of the interval $[0,\infty)$ is not essential: the result immediately generalizes to functions

supported on any half interval $[a, \infty), a \in \mathbb{R}$, simply by replacing the characteristic function $\chi_{[0,\infty)}$ in the translation operator T defined in (2.15) by $\chi_{[a,\infty)}$. Next, if the sequence $\{f_k\}_{k=1}^{\infty}$ can be written as $\{f_k\}_{k=1}^{\infty} = \{g_k\}_{k=1}^{\infty} \cup \{h_k\}_{k=1}^{\infty}$ where supp$g_k \subset [0, \infty)$ and supp $h_k \in (-\infty, L]$ for some $L \in \mathbb{R}$, then a similar procedure as suggested in the particular case of a Hilbert space in [12] can be applied on $\{g_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ separately. In this case, the sequence $\{g_k\}_{k=1}^{\infty} \cup \{h_k\}_{k=1}^{\infty}$ can be approximated with a union of two suborbits, each associated with a bounded operator. We refer the interested reader to [12] for details.

Theorem 2.4 Let X denote a solid translation-invariant Banach function space with absolutely continuous norm, and assume that the translation operator T_1 acts boundedly on X. Let $\lambda > \|T_1\|$ and $\mu > \lambda\|T_{-1}\|$. Assume that $\{f_k\}_{k=1}^{\infty} \subset X$ and supp$f_k \subset [0, \infty)$. Also assume that for every $k \in \mathbb{N}$, there exist $a_k \in \mathbb{N}$ and $C_k > 0$ such that for every $a \geq a_k$,

$$\|f_k \chi_{[a,\infty)}\| \leq C_k \mu^{-a} \tag{2.16}$$

Fixing now a sequence $\{\epsilon_k\}_{k=1}^{\infty} \in \ell^1$ of positive scalars, choose an increasing sequence $\{\alpha(k)\}_{k=1}^{\infty}$ of nonnegative integers such that

$$\|S\|^{\alpha(k)-\alpha(k-1)} \|f_k\| < 1/2\epsilon_k,\ k \geq 2 \tag{2.17}$$

Then $\varphi := \sum_{k=1}^{\infty} S^{\alpha(k)} f_k$ is well-defined. Furthermore, if we also assume that $\alpha(1) = 0$ and

$$\alpha(k+1) - \alpha(k) \geq a_k,\ k \in \mathbb{N} \tag{2.18}$$

and

$$\alpha(k) \geq \frac{\ln(2) + \ln\left(\sum_{n=1}^{k-1} C_n((\lambda\|T_{-1}\|)^{-1}\mu)^{\alpha(n)}\right) - \ln\left(\sum_{j=k+1}^{\infty} \epsilon_j\right)}{\ln((\lambda\|T_{-1}\|)^{-1}\mu)} \tag{2.19}$$

then

$$\|f_k - T^{\alpha(k)}\varphi\| \leq \sum_{n=k+1}^{\infty} \epsilon_n$$

Proof. For every $m, n \in \mathbb{N}$, by (2.17) and since $\|S\| < 1$,

$$\begin{aligned} \left\|\sum_{k=m}^{N} S^{\alpha(k)} f_k\right\| &\leq \sum_{k=m}^{N} \|S\|^{\alpha(k)} \|f_k\| \\ &< 1/2 \sum_{k=m}^{N} \epsilon_k \to 0 \text{ as } m, n \to \infty \end{aligned}$$

As in the proof of Theorem 2.1, it follows that φ is well-defined. Also for $k \in \mathbb{N}$, we have

$$\|f_k - T^{\alpha(k)}\varphi\| \leq \sum_{n=1}^{k-1} \|T^{\alpha(k)} S^{\alpha(n)} f_n\| + \left\|\sum_{n=k+1}^{\infty} T^{\alpha(k)} S^{\alpha(n)} f_n\right\|. \tag{2.20}$$

We now consider the two terms at the right-hand side of the inequality separately. First, using (2.17), we obtain

$$\left\|\sum_{n=k+1}^{\infty} T^{\alpha(k)}S^{\alpha(n)}f_n\right\| = \left\|\sum_{n=k+1}^{\infty} S^{\alpha(n)-\alpha(k)}f_n\right\| \leq \sum_{n=k+1}^{\infty} \left\|S^{\alpha(n)-\alpha(k)}f_n\right\|$$
$$\leq \sum_{n=k+1}^{\infty} \|S\|^{\alpha(n)-\alpha(k)}\|f_n\| \leq \sum_{n=k+1}^{\infty} \|S\|^{\alpha(n)-\alpha(n-1)}\|f_n\|$$
$$\leq 1/2 \sum_{n=k+1}^{\infty} \epsilon_n$$

Next, we get

$$\sum_{n=1}^{k-1} \left\|T^{\alpha(k)}S^{\alpha(n)}f_n\right\| = \sum_{n=1}^{k-1} \left\|T^{\alpha(k)-\alpha(n)}f_n\right\|$$
$$= \sum_{n=1}^{k-1} \left\|\lambda^{\alpha(k)-\alpha(n)}\left(T_{-1}^{\alpha(k)-\alpha(n)}f_n\right)\chi_{[0,\infty)}\right\|$$
$$= \sum_{n=1}^{k-1} \lambda^{\alpha(k)-\alpha(n)}\left\|T_{-1}^{\alpha(k)-\alpha(n)}\left(f_n\chi_{[\alpha(k)-\alpha(n),\infty)}\right)\right\|$$
$$\leq \sum_{n=1}^{k-1} (\lambda\|T_{-1}\|)^{\alpha(k)-\alpha(n)}\left\|f_n\chi_{[\alpha(k)-\alpha(n),\infty)}\right\|$$

Since $n \leq k-1$, we have $\alpha(k)-\alpha(n) \geq \alpha(n+1)-\alpha(n)$ and therefore by (2.18), we get $\alpha(k)-\alpha(n) \geq a_n$. Therefore, by (2.16),

$$\sum_{n=1}^{k-1} \left\|T^{\alpha(k)}S^{\alpha(n)}f_n\right\| \leq \sum_{n=1}^{k-1} (\lambda\|T_{-1}\|)^{\alpha(k)-\alpha(n)}C_n\mu^{-(\alpha(k)-\alpha(n))}$$

Now if $\alpha(k)$ is chosen such that (2.19) holds, we have that
$\alpha(k)\ln\left((\lambda\|T_{-1}\|)^{-1}\mu\right) \geq \ln(2) + \ln\left(\sum_{n=1}^{k-1} C_n((\lambda\|T_{-1}\|)^{-1}\mu)^{\alpha(n)}\right) - \ln\left(\sum_{j=k+1}^{\infty}\epsilon_j\right)$
or
$\alpha(k)\ln\left(\lambda\|T_{-1}\|\mu^{-1}\right) + \ln\left(\sum_{n=1}^{k-1} C_n((\lambda\|T_{-1}\|)^{-1}\mu)^{\alpha(n)}\right) \leq \ln\left(1/2\sum_{j=k+1}^{\infty}\epsilon_j\right)$.
Applying the exponential function on both sides of the inequality yields

$$(\lambda\|T_{-1}\|\mu^{-1})^{\alpha(k)} \sum_{n=1}^{k-1} C_n((\lambda\|T_{-1}\|)^{-1}\mu)^{\alpha(n)} \leq 1/2 \sum_{j=k+1}^{\infty} \epsilon_j,$$

and from (2.21), we conclude that

$$\sum_{n=1}^{k-1} \left\|T^{\alpha(k)}S^{\alpha(n)}f_n\right\| \leq 1/2 \sum_{j=k+1}^{\infty} \epsilon_j$$

Using now (2.20) and the obtained estimates of the two terms, we obtain the desired result.

In the next example we consider an important class of Banach spaces that satisfy all the conditions in Theorem 2.4.

Example 2.5 Let $m: \mathbb{R} \to [0,\infty)$ be a continuous function and $w: \mathbb{R} \to [0,\infty)$ a m-moderate weight function, i.e., a measurable function such that

$$w(x+y) \leq m(x)w(y),\ \forall x,y \in \mathbb{R}$$

For $1 \leq p < \infty$, let

$$L_w^p(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{C} \left| \int_{\mathbb{R}} \left| f(x) \right|^p w(x)dx < \infty \right. \right\}$$

Then $L_w^p(\mathbb{R})$ is a Banach space with respect to the norm

$$\|f\|_{L_w^p} = \left(\int_{\mathbb{R}} |f(x)|^p w(x)dx \right)^{1/p}$$

We leave it to the reader to verify that the norm is absolutely continuous, $L_w^p(\mathbb{R})$ is invariant under the translation-operators T_1, T_{-1}, and that $\|T_{-1}\| \leq m(-1)^{1/p}$ and $\|T_1\| \leq m(1)^{1/p}$.

In order to demonstrate the practical issues showing up in applications of Theorem 2.4, we will now consider so-called Gabor systems. For $b \in \mathbb{R}$, let $E_b : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the modulation operator defined as $E_b g(x) = e^{2\pi ibx} g(x),\ x \in \mathbb{R}$. Fixing a function $g \in L^2(\mathbb{R})$ and some parameters $a, b > 0$, the sequence of functions $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$ is called a Gabor system. Since the translation operators and modulation operators clearly act boundedly on any L^p-space, $1 \leq p < \infty$ as well, we will now assume that $g \in L^p(\mathbb{R})$ and consider the Gabor system in $L^p(\mathbb{R})$ instead. Note that if the function g is compactly supported, we can split the Gabor system into a union $\{g_k\}_{k=1}^{\infty} \cup \{h_k\}_{k=1}^{\infty}$ where supp$g_k \subset [0,\infty)$ and supp$h_k \in (-\infty, L]$ for some $L \in \mathbb{R}$; thus, as explained just before the statement of Theorem 2.4 we can approximate the Gabor system using suborbits of two bounded operators. In the next example we will replace the assumption that g has compact support by the assumption that supp $g \subset [0,\infty)$, and show how to obtain the estimate (2.16) for a certain ordering of a the "half Gabor system" $\{E_{mb}T_{na}g\}_{m\in\mathbb{Z},n\in\mathbb{N}\cup\{0\}}$.

Example 2.6 Consider a function $g \in L^p(\mathbb{R}), 1 \leq p < \infty$, and assume that supp $g \subset [0,\infty)$ and that there exist constants $C, d_0 > 0$ and $\mu > 1$ such that for all $d \geq d_0$

$$\left(\int_d^{\infty} |g(x)|^p dx \right)^{1/p} \leq C\mu^{-d} \tag{2.22}$$

Re-index the "half Gabor system" $\{E_{mb}T_{na}g\}_{m\in\mathbb{Z},n\in\mathbb{N}\cup\{0\}}$ as $\{f_k\}_{k=1}^{\infty}$ in such a way that f_k and f_{k+1} differ with at most one translate by a, i.e., if $f_k = E_{mb}T_{na}g$ for some $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ then f_{k+1} is one of the following functions

$$E_{mb}T_{(n\pm1)a}g,\ E_{(m\pm1)b}T_{na}g$$

Now, for $k \in \mathbb{N}$, write $f_k = E_{mb}T_{na}g$, where $n \in \{0,1,\ldots,k\}$ and $m \in \mathbb{Z}$. Then, considering any $d > 0$, we have

$$\int_d^{\infty} |f_k(x)|^p dx = \int_d^{\infty} |E_{mb}T_{na}g(x)|^p dx = \int_d^{\infty} |g(x-na)|^p dx = \int_{d-na}^{\infty} |g(x)|^p dx$$

For $d \geq d_0 + ka$, using that $n \in \{0,1,\ldots,k\}$, (2.22) yields

$$\left(\int_d^{\infty} |f_k(x)|^p dx \right)^{1/p} \leq C\mu^{-(d'-na)} \leq (C\mu^{ka})\mu^{-d}$$

Thus, choosing $C_k := (C\mu^{ka})$ and $d_k := d_0 + ka$, we have that for any $d \geq d_k$,

$$\left\| f_k \chi_{[d,\infty)} \right\|_p \leq C_k \mu^{-d} \tag{2.23}$$

i.e., the condition (2.22) is satisfied. By a direct calculation, the "new technical assumption" (2.22) holds if, e.g.,

$$|g(x)| \leq e^{-x}\chi_{[0,\infty)}, \ \forall x \in \mathbb{R}$$

indeed, in this we can take $C = 1/p$ and $\mu = e^p$. Note that for $p = 2$, the function $h(x) = e^{-x}\chi_{[0,\infty)}$ play a special role in Gabor analysis: it generates a Gabor frame $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$ if and only if $ab \leq 1$, see [27].

2.4 ϵ-close approximations of $\{f_k\}_{k=1}^{\infty}$

In this section we will pave the way for the application of the theoretical results in Section 2.5. To motivate what follows, consider again Theorem 2.1: for any given finite sequence $\{f_k\}_{k=1}^{\infty}$ in X and any sequence $\{\epsilon_k\}_{k=1}^{\infty} \in \ell^1$ of positive scalars it specifies how to choose $\varphi \in X$ and powers $\alpha(k)$ such that for each $k \in \mathbb{N}$ the vector $T^{\alpha(k)}\varphi$ belongs to a ball around f_k, with a radius specified by (2.5). The goal of typical approximation arguments is that the stated condition should imply that the approximating sequence - here $\{T^{\alpha(k)}\varphi\}_{k=1}^{\infty}$ shares key features of the given sequence $\{f_k\}_{k=1}^{\infty}$. In concrete cases this might call for extra conditions on the sequence $\{\epsilon_k\}_{k=1}^{\infty} \in \ell^1$. Recall (see, e.g., [29]) that given a sequence $\{f_k\}_{k=1}^{\infty}$ in a Hilbert space $\mathcal{H}$, a sequence $\{g_k\}_{k=1}^{\infty} \subset \mathcal{H}$ is said to be quadratically close to $\{f_k\}_{k=1}^{\infty}$ if

$$\sum_{k=1}^{\infty} \|f_k - g_k\|^2 < 1$$

This concept is well-motivated. Indeed, if $\{f_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ and $\{g_k\}_{k=1}^{\infty} \subset \mathcal{H}$ is quadratically close to $\{f_k\}_{k=1}^{\infty}$, then $\{g_k\}_{k=1}^{\infty}$ is a Riesz basis for $\mathcal{H}$; see [29]. The following generalizes the concept of quadratically close sequences.

Definition 2.7 Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in a Banach space X. Given any $p \in [1,\infty)$ and any $\epsilon > 0$, a sequence $\{g_k\}_{k=1}^{\infty} \subset X$ is said to be ϵ-close to $\{f_k\}_{k=1}^{\infty}$ with respect to ℓ^p if

$$\sum_{k=1}^{\infty} \|f_k - g_k\|^p \leq \epsilon \tag{2.24}$$

In the setting of Theorem 2.1, Theorem 2.2 or Theorem 2.4 we will now show how to ensure that the constructed sequence $\{T^{\alpha(k)}\varphi\}_{k=1}^{\infty}$ is ϵ-close to the given sequence $\{f_k\}_{k=1}^{\infty}$. Recall that on the structural level, the three mentioned results are similar: all of them ensure that for a given sequence $\{f_k\}_{k=1}^{\infty}$ in the considered Banach space and a fixed sequence $\{\epsilon_j\}_{j=1}^{\infty} \in \ell^1$, the operator T, the vector φ, and the powers $\alpha(k)$ satisfy the inequality (2.1).

Corollary 2.8 Let $\{f_k\}_{k=1}^{\infty} \subset X$. Fix a positive sequence $\{\epsilon_j\}_{j=1}^{\infty} \in \ell^1$, and assume that the operator $T: X \to X, \varphi \in X$, and powers $\alpha(k), k \in \mathbb{N}$, have been constructed such that (2.1) is satisfied. Then the following holds true:
(i) Let $\epsilon_j := \epsilon 2^{-j}$. Then the sequence $\{T^{\alpha(k)}\varphi\}_{k=1}^{\infty}$ is ϵ-close to $\{f_k\}_{k=1}^{\infty}$ simultaneously for all $p \in [1,\infty)$. More precisely, it holds that for all $p \geq 1$,

$$\sum_{k=1}^{\infty} \left\|f_k - T^{\alpha(k)}\varphi\right\|^p \leq \frac{\epsilon^p}{2^p - 1} \leq \epsilon$$

(ii) Fix any $p \in [1,\infty)$ and consider a weight sequence $\{w_k\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} \frac{1}{2^{kp}} w_k < \infty$. Let $M := \max\left\{1, \sum_{k=1}^{\infty} \frac{1}{2^{kp}} w_k\right\}$ and $\epsilon_j := \frac{\epsilon}{M} 2^{-j}$. Then

$$\sum_{k=1}^{\infty} \left\|f_k - T^{\alpha(k)}\varphi\right\|^p w_k \le \epsilon^p$$

(iii) Consider a solid Banach sequence space X_d for which $\{2^{-k}\}_{k=1}^{\infty} \in X_d$. Take $M := \left\|\{2^{-k}\}_{k=1}^{\infty}\right\|_{X_d}$ and let $\epsilon_j := \epsilon/M2^{-j}$. Then

$$\left\|\{\|f_k - T^{\alpha(k)}\varphi\|\}_{k=1}^{\infty}\right\|_{X_d} \le \epsilon \tag{2.25}$$

Proof. For the proof of (i), just observe that

$$\sum_{k=1}^{\infty} \left\|f_k - T^{\alpha(k)}\varphi\right\|^p \le \sum_{k=1}^{\infty}\left(\sum_{j=k+1}^{\infty} \frac{\epsilon}{2^j}\right)^p = \epsilon^p \sum_{k=1}^{\infty}\left(\frac{1}{2^k}\right)^p = \frac{\epsilon^p}{2^p - 1}$$

Similarly, under the assumptions in (ii),

$$\begin{aligned}\sum_{k=1}^{\infty} \left\|f_k - T^{\alpha(k)}\varphi\right\|^p w_k &\le \sum_{k=1}^{\infty}\left(\sum_{j=k+1}^{\infty} \epsilon/M2^{-j}\right)^p w_k \\ &= \sum_{k=1}^{\infty} (\epsilon/M)^p \frac{1}{2^{kp}} w_k \\ &\le \epsilon^p M^{1-p} \le \epsilon^p\end{aligned}$$

For the proof of (iii), Theorem 2.1 gives

$$\left\|f_k - T^{\alpha(k)}\varphi\right\| \le \epsilon/M2^{-k} \tag{2.26}$$

Since X_d is a solid Banach sequence space and $\{2^{-k}\}_{k=1}^{\infty} \in X_d$, this implies $\left\{\left\|f_k - T^{\alpha(k)}\varphi\right\|\right\}_{k=1}^{\infty} \in X_d$ and that

$$\left\|\{\|f_k - T^{\alpha(k)}\varphi\|\}_{k=1}^{\infty}\right\|_{X_d} \le \left\|\{\epsilon/M2^{-k}\}_{k=1}^{\infty}\right\|_{X_d} = \epsilon$$

as claimed.

2.5 Applications to atomic decompositions

We will now apply the theoretical results to atomic decompositions in Banach spaces. Let us first state the definition:

Definition 2.9 Consider a Banach space X, a Banach sequence space X_d, and two arbitrary sequences $\{e_k\}_{k=1}^{\infty} \subset X$ and $\{e_k^*\}_{k=1}^{\infty} \in X^*$. The pair $(\{e_k\}_{k=1}^{\infty}, \{e_k^*\}_{k=1}^{\infty})$ is called an atomic decomposition of X with respect to X_d, with bounds $A, B > 0$, if
(i) $\{\langle x, e_k^*\rangle\}_{k=1}^{\infty} \in X_d$ for all $x \in X$;
(ii) $A\|x\| \le \|\{\langle x, e_k^*\rangle\}_{k=1}^{\infty}\|_{X_d} \le B\|x\|$ for all $x \in X$;
(iii) $x = \sum_{k=1}^{\infty} \langle x, e_k^*\rangle e_k$ for all $x \in X$.

Note that if $\mathcal{H}$ is a separable Hilbert space and $X_d = \ell^2$, the conditions (i)+(ii) automatically imply the existence of a sequence $\{e_k\}_{k=1}^{\infty} \in \mathcal{H}$ such that (iii) holds. This is well-studied in the literature on frames, see, e.g., [9]. On the other hand, in Banach spaces the so-called reconstruction property in (iii) does not follow from (i)+(ii) and has to be assumed separately. Regardless whether (i) + (ii) holds or not, the reconstruction property (iii) clearly holds if $\{e_k\}_{k=1}^{\infty}$ is a basis for the Banach space X and $\{e_k^*\}_{k=1}^{\infty}$ is the dual basis. "Genuine" atomic decompositions have been constructed, e.g., in [18, 21, 22, 30]. Stability conditions for atomic decompositions were studied in the paper [16]; more precisely, it was shown that if a sequence $\{f_k\}_{k=1}^{\infty} \subset X$ yields an atomic decomposition with respect to a sequence $\{g_k\}_{k=1}^{\infty} \subset X^*$, then a sufficiently small perturbation $\{f_k'\}_{k=1}^{\infty}$ of $\{f_k\}_{k=1}^{\infty}$ also yields an atomic decomposition with respect to a certain sequence $\{g_k'\}_{k=1}^{\infty}$. The following result specifies how to ensure that the perturbation condition considered in [16] is satisfied in the current setting:

Corollary 2.10 Assume that $(\{f_k\}_{k=1}^{\infty}, \{g_k\}_{k=1}^{\infty})$ is an atomic decomposition of X with respect to ℓ^p for some $p \in [1,\infty)$, with bounds A, B. Take $\epsilon_j = \epsilon 2^{-j}$ for some $0 < \epsilon < B^{-1}$. Then, if (2.1) holds, there exists a family $\{g_k'\}_{k=1}^{\infty} \in X^*$ such that $\left(\{T^{\alpha(k)}\varphi\}_{k=1}^{\infty}, \{g_k'\}_{k=1}^{\infty}\right)$ is an atomic decomposition of X with respect to ℓ^p, with bounds $A(1+\epsilon B)^{-1}$ and $B(1-\epsilon B)^{-1}$. Moreover $\{T^{\alpha(k)}\varphi\}_{k=1}^{\infty}$ is a basis for X if and only if $\{f_k\}_{k=1}^{\infty}$ is a basis for X.

Proof. Corollary 2.8 (i) implies that if $\{f_k\}_{k=1}^{\infty}$ satisfies the conditions stated in either Theorem 2.1, Theorem 2.2 or Theorem 2.4 then for every finite sequence $\{c_k\}$,

$$\begin{aligned}\left\|\sum c_k\left(f_k - T^{\alpha(k)}\varphi\right)\right\|_X &\leq \sum |c_k| \left\|f_k - T^{\alpha(k)}\varphi\right\|_X \\ &\leq \left(\sum |c_k|^p\right)^{1/p} \left(\sum \left\|f_k - T^{\alpha(k)}\varphi\right\|_X^q\right)^{1/q} \\ &\leq \epsilon \|\{c_k\}\|.\end{aligned}$$

The obtained inequality is a special case of the condition in Theorem 2.3 in [16], which immediately yields the stated conclusion.

A more general way of obtaining decompositions in Banach spaces is obtained by considering Banach frames rather than atomic decompositions. While an atomic decomposition reconstructs an element $x \in X$ using an infinite linear combination of the vectors $\{e_k\}_{k=1}^{\infty}$ with coefficients $\langle x, e_k^*\rangle, k \in \mathbb{N}$, the definition of a Banach frame ensures the existence of a certain reconstruction operator, which maps the coefficients $\langle x, e_k^*\rangle, k \in \mathbb{N}$, back to the vector $x \in X$. Again based on a stability result from the paper [16] and in a completely similar fashion as the proof of Corollary 2.10, one can prove that if a sequence $\{f_k\}_{k=1}^{\infty} \subset X^*$ generates a Banach frame with respect to a certain bounded operator, then an estimate of the type (2.1) implies that also the sequence $\{T^{\alpha(k)}\varphi\}_{k=1}^{\infty}$ generates a Banach frame. Due to the similarity with Corollary 2.10 we will not go into details, but just stress the fact that due to the setting of Banach frames the application of our theoretical results will take place in the dual space X^* for this particular case.

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