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Identifying Restrained Domination in Graphs

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Abstract
Let G be a connected simple graph. A subset S of $V(G)$ is a dominating set of G if for every $v \in V(G) \setminus S$, there exists $x \in S$ such that $xv \in E(G)$. An identifying code S of a graph G is a dominating set $S \subseteq V(G)$ such that for every $v \in V(G)$, $N_G[v] \cap S$ is distinct. An identifying code of a graph G is an identifying restrained dominating set if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. Alternately, an identifying code of a graph $S \subseteq V(G)$ is an identifying restrained dominating set if $N[S] = V(G)$ and $\langle V(G) \setminus S \rangle$ is a subgraph without isolated vertices. The minimum cardinality of an identifying restrained dominating set of G, denoted by $\gamma_r^{ID}(G)$, is called the identifying restrained domination number of G. In this paper, we initiate the study of the concept and give the domination number of some special graphs. Further, we show the characterization of the identifying restrained dominating set in the join of two nontrivial connected graphs.

Keywords: dominating set, identifying code, restrained dominating set, identifying restrained dominating set

1. Introduction

Domination in graph theory was introduced by Claude Berge in 1958 and Oystein Ore in 1962 [1]. Following an article [2] by Ernie Cockayne and Stephen Hedetniemi in 1977, the domination in graphs became an area of study by many researchers. A subset S of $V(G)$ is a dominating set of G if for every $v \in V(G) \setminus S$, there exists $x \in S$ such that $xv \in E(G)$, i.e. $N[S] = V(G)$. The domination number $\gamma(G)$ of G is the smallest cardinality of a dominating set of G. Some studies on domination in graphs were found in the papers [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17].

The identifying code of a graph was studied in 1998 by M.G. Karpovsky et al. [18] in their paper "On a new class of codes for identifying vertices in graphs.". They observed that the concept of identifying codes is that a graph is identifiable if and only if it is twin-free. A vertex x is a twin of another vertex y if $N[x] = N[y]$. A graph G is called twin-free if no vertex has a twin. An identifying code of a graph G is a dominating set $C \subseteq V(G)$ such that for every $v \in V(G)$, $N_G(v) \cap C$ is distinct. The minimum cardinality of an identifying code of G, denoted by $\gamma^{ID}(G)$, is called the identifying code number of G. An identifying code of cardinality $\gamma^{ID}(G)$ is called an γ^{ID}-set of G. From a computational point of view, it is shown that given a graph G, finding the exact value of $\gamma^{ID}(G)$ is in the class of NP-hard problems. It in fact remains NP-hard for many subclasses of graphs [19, 20]. Furthermore, approximating $\gamma^{ID}(G)$ is not easy as shown in [21, 22, 23]. Identifying code of a graph is also studied in [24].

The restrained domination in graphs was introduced by Telle and Proskurowski [26] indirectly as a vertex partitioning problem. Accordingly, a set $S \subseteq V(G)$ is a restrained dominating set if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. Alternately, a subset S of $V(G)$ is a restrained dominating set if $N[S] = V(G)$ and $\langle V(G) \setminus S \rangle$ is a subgraph without isolated vertices. The minimum cardinality of a restrained dominating set of G, denoted by $\gamma_r(G)$, is called the restrained domination number of G. A restrained dominating set of cardinalities $\gamma_r(G)$ is called an γ_r-set. Restrained domination in graphs was also found in the papers [27, 28, 29, 30, 31, 32, 33, 34, 35, 36].

The identifying code in graphs and restrained domination in graphs have motivated the researchers to introduce a new domination in graphs – an identifying restrained domination in graphs. An identifying code S of a graph G is an identifying restrained dominating set if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. Alternately, an identifying code of a graph $S \subseteq V(G)$ is an identifying restrained dominating set if $N[S] = V(G)$ and $\langle V(G) \setminus S \rangle$ is a subgraph without isolated vertices. The minimum cardinality of an identifying restrained dominating set of G, denoted by $\gamma_r^{ID}(G)$, is called the identifying restrained domination number of G. In this paper, we initiate the study of the concept and give the domination number of some special graphs. Further, we show the characterization of the identifying restrained dominating set in the join of two nontrivial connected graphs.

For the general terminology in graph theory, readers may refer to [37]. A graph G is a pair $\big(V(G), E(G)\big)$, where $V(G)$ is a finite nonempty set called the vertex-set of G and $E(G)$ is a set of unordered pairs $\{u, v\}$ (or simply uv) of distinct elements from $V(G)$ called the edge-set of G. The elements of $V(G)$ are called vertices and the cardinality $|V(G)|$ of $V(G)$ is the order of G. The elements of $E(G)$ are called edges and the cardinality $|E(G)|$ of $E(G)$ is the size of G. If $|V(G)| = 1$, then G is called a trivial graph. If $|E(G)| = \emptyset$, then G is called an empty graph. The open neighborhood of a vertex $v \in V(G)$ is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The elements of $N_G(v)$ are called neighbors of v. The closed neighborhood of $v \in V(G)$ is the set $N_G[v] = N_G(v) \cup \{v\}$. If $X \subseteq V(G)$, the open neighborhood of X in G is the set $N_G(X) = \bigcup_{v \in X} N_G(v)$. The closed neighborhood of X in G is the set $N_G[X] = \bigcup_{v \in X} N_G[v]$.

2. Results

Definition 2.1. [1] Let G be a directed or undirected graph with the vertex set $V(G)$. A subset D of $V(G)$ is a dominating set for G when every vertex not in D is the endpoint of some edge from a vertex in D. Clearly, $V(G)$ itself is a dominating set. A minimum dominating set is a dominating set such that no subset has this property. The domination number $\gamma(G)$ of a graph G is the smallest number of vertices in any minimum dominating set.

Definition 2.2. [24] An identifying code of a graph G is a dominating set $C \subseteq V(G)$ such that for every vertex $v \in V(G)$, $N_G[v] \cap C$ is distinct. The minimum cardinality of an identifying code of G, denoted by $\gamma^{ID}(G)$, is called the identifying code number of G. An identifying code of cardinality $\gamma^{ID}(G)$ is called a γ^{ID}-set of G.

Definition 2.3. [38] Let $G = (V, E)$ be a graph. A restrained dominating set is a set $S \subseteq V$ where every vertex in $V - S$ is adjacent to a vertex in S as well as another vertex in $V - S$. Alternately, a subset S of $V(G)$ is a restrained dominating set if $N[S] = V(G)$ and $\langle V(G) \setminus S \rangle$ is a subgraph without isolated vertices.

The restrained domination number of G, denoted by $\gamma_r(G)$, is the smallest cardinality of a restrained dominating set of G.

Remark 2.4. Every graph G has a restrained dominating set, since V(G) is such a set.

Definition 2.5. A dominating set S of vertices of a graph G is an identifying restrained dominating set of G if S is a restrained dominating set and for every two vertices x and y, the sets $N_G[x] \cap S$ and $N_G[y] \cap S$ are nonempty and distinct. The identifying restrained domination number of G, denoted by $\gamma_r^{ID}(G)$, is the minimum cardinality of an identifying restrained dominating set of G. An identifying restrained dominating set of cardinality $\gamma_r^{ID}(G)$ will be called γ_r^{ID}-set.

Remark 2.6. Let G be a non-complete graph. If $S \subseteq V(G)$ is both an identifying code and a restrained dominating set of G, then S is an identifying restrained dominating set of G.

Proposition 2.7. Let $G = P_n$ for all integers $n \geq 8$. Then

$$\gamma_r^{ID}(G) = \begin{cases} \frac{3n}{5} + 2, & \text{if } n \equiv 0 \pmod 5, \\ \frac{3n+12}{5}, & \text{if } n \equiv 1 \pmod 5, \\ \frac{3n+14}{5}, & \text{if } n \equiv 2 \pmod 5, \\ \frac{3n+6}{5}, & \text{if } n \equiv 3 \pmod 5, \\ \frac{3n+8}{5}, & \text{if } n \equiv 4 \pmod 5. \end{cases}$$

We need the following results for our subsequent Theorem and Corollary.

Lemma 2.8. Let G and H be connected non-complete graphs. If S_G is an identifying code of G and S_H is an identifying code of H with $\gamma(\langle S_G \rangle) \neq 1$ and $\gamma(\langle S_H \rangle) \neq 1$, then $S_G \cup S_H$ is an identifying code of $G + H$.

Proof. Suppose that S_G is an identifying code of G and S_H is an identifying code of H with $\gamma(\langle S_G \rangle) \neq 1$ and $\gamma(\langle S_H \rangle) \neq 1$. Consider that $S = S_G \cup S_H$.

Case 1. If $v, v' \in V(G)$ and $v \neq v'$, then $N_G[v] \cap S_G \neq N_G[v'] \cap S_G$. Since $v, v' \in V(G) \subset V(G+H)$, it follows that,

$$\begin{aligned} N_{G+H}[v] \cap S &= N_{G+H}[v] \cap (S_G \cup S_H) \\ &= (N_{G+H}[v] \cap S_G) \cup (N_{G+H}[v] \cap S_H) \\ &= (N_{G+H}[v] \cap S_G) \cup S_H \\ &= (N_G[v] \cap S_G) \cup S_H \\ &\neq (N_G[v'] \cap S_G) \cup S_H \\ &= (N_{G+H}[v'] \cap S_G) \cup S_H \\ &= (N_{G+H}[v'] \cap S_G) \cup (N_{G+H}[v'] \cap S_H) \\ &= N_{G+H}[v'] \cap (S_G \cup S_H) \\ &= N_{G+H}[v'] \cap S. \end{aligned}$$

Case 2. If $u, u' \in V(H)$ and $u \neq u'$, then $N_H[u] \cap S_H \neq N_H[u'] \cap S_H$. Since $u, u' \in V(H) \subset V(G+H)$, it follows that,

$$\begin{aligned} N_{G+H}[u] \cap S &= N_{G+H}[u] \cap (S_G \cup S_H) \\ &= (N_{G+H}[u] \cap S_G) \cup (N_{G+H}[u] \cap S_H) \\ &= S_G \cup (N_{G+H}[u] \cap S_H) \\ &= S_G \cup (N_H[u] \cap S_H) \\ &\neq S_G \cup (N_H[u'] \cap S_H) \\ &= S_G \cup (N_{G+H}[u'] \cap S_H) \\ &= (N_{G+H}[u'] \cap S_G) \cup (N_{G+H}[u'] \cap S_H) \\ &= N_{G+H}[u'] \cap (S_G \cup S_H) \\ &= N_{G+H}[u'] \cap S. \end{aligned}$$

Case 3. If $v \in V(G)$ and $u \in V(H)$, then $(N_G[v] \cap S_G) \cup S_H \neq S_G \cup (N_H[u] \cap S_H)$ because $\gamma(\langle S_G \rangle) \neq 1$ and $\gamma(\langle S_H \rangle) \neq 1$. Since $v \in V(G) \subset V(G+H)$ and $u \in V(H) \subset V(G+H)$, it follows that,

$$\begin{aligned} N_{G+H}[v] \cap S &= N_{G+H}[v] \cap (S_G \cup S_H) \\ &= (N_{G+H}[v] \cap S_G) \cup (N_{G+H}[v] \cap S_H) \\ &= (N_G[v] \cap S_G) \cup S_H \\ &\neq S_G \cup (N_H[u] \cap S_H) \\ &= (N_{G+H}[u] \cap S_G) \cup (N_{G+H}[u] \cap S_H) \\ &= N_{G+H}[u] \cap (S_G \cup S_H) \\ &= N_{G+H}[u] \cap S. \end{aligned}$$

Thus, by referring to Case 1, Case 2, and Case 3, for all $u, v \in V(G+H)$, $N_{G+H}[u] \cap S \neq N_{G+H}[v] \cap S$. This implies that for every vertex $v \in V(G+H)$, $N_{G+H}[v] \cap S$ is distinct. Hence, $S = S_G \cup S_H$ is an identifying code in $G+H$ by Definition 2.2.

Lemma 2.9. Let $G = P_n = [v_1, v_2, \dots, v_n], n = 2k+1$ for all positive integer k. Then $S = \{v_1, v_3, v_5, \dots, v_n\}$ is a γ^{ID}-set of G.

Proof. Let $v_k \in V(G)$ where $k \in \{1,2,3, \dots, n\}$.

Case 1. Consider that k is an odd integer.

Subcase 1. If $k = 1$, then $N_G[v_1] \cap S = \{v_1, v_2\} \cap \{v_1, v_3, v_5, \dots, v_n\} = \{v_1\}$.
Subcase 2. If $k = n$, then $N_G[v_n] \cap S = \{v_{n-1}, v_n\} \cap \{v_1, v_3, v_5, \dots, v_n\} = \{v_n\}$.
Subcase 3. If $k \neq 1$ and $k \neq n$, then

$$\begin{aligned} N_G[v_k] \cap S &= \{v_{k-1}, v_k, v_{k+1}\} \cap \{v_1, v_3, v_5, \dots, v_{k-2}, v_k, v_{k+2}, \dots, v_n\} \\ &= \{v_k\}. \end{aligned}$$

By Subcase 1, Subcase 2, and Subcase 3, the $N_G[v_k] \cap S$ is distinct for all odd integer $k \in \{1,2,3, \dots, n\}$.

Case 2. Consider that k is an even integer. Then

$$\begin{aligned} N_G[v_k] \cap S &= \{v_{k-1}, v_{k+1}\} \cap \{v_1, v_3, v_5, \dots, v_{k-1}, v_{k+1}, \dots, v_n\} \\ &= \{v_{k-1}, v_{k+1}\}. \end{aligned}$$

Thus, $N_G[v_k] \cap S$ is distinct for all even integer $k \in \{1,2,3,\dots,n\}$.

Therefore, $N_G[v_k] \cap S$ is distinct for all integer $k \in \{1,2,3,\dots,n\}$ by combining Case 1 and Case 2. By using the definition, S is an identifying code in G.

Suppose that S is not a minimum identifying code in G. Then, there exists a minimum identifying code S' such that $|S'| < |S|$. If $S' = S \setminus \{v\}$ for some $v \in S$, then S' is not a dominating set in G. If $S' = V(G) \setminus S = \{v_2, v_4, \dots, v_{n-1}\}$, then $N_G[v_1] \cap S' = N_G[v_2] \cap S' = \{v_2\}$. That is, $N_G[v_k] \cap S'$ is not distinct for some positive integer k. Hence, S' is not an identifying code in G. If S' is any other identifying code not equal to S, then $|S'| = |S|$ if $G = P_5$, otherwise $|S'| > |S|$. This contradict to our assumption that S is not a minimum identifying code in G. Therefore, S must be a minimum identifying code in G. Hence, S is a γ^{ID}-set of G. ■

Lemma 2.10. Let $G = P_n = [v_1, v_2, \dots, v_n], n = 2k + 2$ for all positive integer $k \geq 3$. Then $S = \{v_1, v_3, v_5, \dots, v_{n-5}, v_{n-3}, v_{n-2}, v_{n-1}\}$ is a γ^{ID}-set of G.

Proof. Let $v_i \in V(G)$ where $i \in \{1,2,3,\dots,n\}$.

Case 1. Consider that i is an odd integer.

Subcase 1. If $i = 1$, then

$$N_G[v_1] \cap S = \{v_1, v_2\} \cap \{v_1, v_3, v_5, \dots, v_{n-5}, v_{n-3}, v_{n-2}, v_{n-1}\} = \{v_1\}.$$

Subcase 2. If $i = n - 1$, then

$$N_G[v_{n-1}] \cap S = \{v_{n-2}, v_{n-1}, v_n\} \cap \{v_1, v_3, v_5, \dots, v_{n-5}, v_{n-3}, v_{n-2}, v_{n-1}\} = \{v_{n-2}, v_{n-1}\}.$$

Subcase 3. If $i \neq 1$ and $i \neq n - 1$, then

$$\begin{aligned} N_G[v_i] \cap S &= \{v_{i-1}, v_i, v_{i+1}\} \cap \{v_1, v_3, v_5, \dots, v_{i-2}, v_i, v_{i+2}, \dots, v_{n-5}, v_{n-3}, v_{n-2}, v_{n-1}\} \\ &= \{v_i\}, if\ v_i \in \{v_3, v_5, \dots, v_{n-5}\}\ or, \end{aligned}$$

$$\begin{aligned} N_G[v_{n-3}] \cap S &= \{v_{n-4}, v_{n-3}, v_{n-2}\} \cap \{v_1, v_3, v_5, \dots, v_{i-2}, v_i, v_{i+2}, \dots, v_{n-5}, v_{n-3}, v_{n-2}, v_{n-1}\} \\ &= \{v_{n-3}, v_{n-2}\}. \end{aligned}$$

By Subcase 1, Subcase 2, and Subcase 3, the $N_G[v_i] \cap S$ is distinct for all odd integer $i \in \{1,2,3,\dots,n\}$.

Case 2. Consider that i is an even integer.

Subcase 1. If $i \in \{2,4,6,\dots,n-4\}$, then

$N_G[v_i] \cap S = \{v_{i-1}, v_i, v_{i+1}\} \cap \{v_1, v_3, v_5, \dots, v_{i-1}, v_{i+1}, \dots, v_{n-5}, v_{n-3}, v_{n-2}, v_{n-1}\} = \{v_{i-1}, v_{i+1}\}$.
Subcase 2. If $i = n-2$, then

$$N_G[v_{n-2}] \cap S = \{v_{n-3}, v_{n-2}, v_{n-1}\} \cap \{v_1, v_3, v_5, \dots, v_{n-5}, v_{n-3}, v_{n-2}, v_{n-1}\} = \{v_{n-3}, v_{n-2}, v_{n-1}\}.$$

Subcase 3. If $i = n$, then

$$\begin{aligned} N_G[v_n] \cap S &= \{v_{n-1}, v_n\} \cap \{v_1, v_3, v_5, \dots, v_{i-2}, v_i, v_{i+2}, \dots, v_{n-5}, v_{n-3}, v_{n-2}, v_{n-1}\} \\ &= \{v_{n-1}\}. \end{aligned}$$

By Subcase 1, Subcase 2, and Subcase 3, the $N_G[v_i] \cap S$ is distinct for all even integer $i \in \{1,2,3, \dots, n\}$. Therefore, $N_G[v_k] \cap S$ is distinct for all integer $k \in \{1,2,3, \dots, n\}$ by combining Case 1 and Case 2. By using the definition, S is an identifying code in G.
Suppose that S is not a minimum identifying code in G. Then, there exists a minimum identifying code S' in G such that $|S'| < |S|$. If $S' = S \setminus \{v\}$ for some $v \in S \setminus \{v_{n-3}, v_{n-2}\}$, then S' is not a dominating set in G. If $S' = V(G) \setminus \{v_{n-2}\}$, then $N_G[v_{n-1}] \cap S' = N_G[v_n] \cap S' = \{v_{n-1}\}$. That is, $N_G[v_i] \cap S'$ is not distinct for some positive integer i. Hence, S' is not an identifying code in G. If S' is any other identifying code not equal to S, then $|S'| > |S|$ for any even integers $n > 6$, the order of G. This contradict to our assumption that S is not a minimum identifying code in G. Therefore, S must be a minimum identifying code in G. Hence, S is a γ^{ID}-set of G.
Theorem 2.11. Let G and H be connected non-complete graphs. The subset $S = S_G \cup S_H \subseteq V(G)$ is an identifying restrained dominating set of $G + H$, if the following conditions are satisfied.
(i) S_G is an identifying code in G or S_G is an identifying restrained dominating set in G,
(ii) S_H is an identifying code in H or S_H is an identifying restrained dominating set in H, and
(iii)$\gamma(\langle S_G \rangle) \neq 1$ and $\gamma(\langle S_H \rangle) \neq 1$.
Proof. Suppose that statements (i), (ii), and (iii) are satisfied. Consider the following cases.
Case 1. If $S_G = V(G)$ and $S_H = V(H)$, then $S = S_G \cup S_H = V(G + H)$. By hypothesis (i) and (ii), and the Lemma 2.8, S is an identifying code of $G + H$. Since $V(G + H)$ is a restrained dominating set, by Remark 2.4, it follows that $S = V(G + H)$ is an identifying restrained dominating set of $G + H$ by Remark 2.6.
Case 2. If $S_G = V(G)$ and $S_H \subset V(H)$, then in view of the hypothesis (ii), S_H is an identifying restrained dominating set in H. Further, $S = S_G \cup S_H$ is an identifying code of $G + H$ by hypothesis and by Lemma 2.8. Let $u, u' \in V(H) \setminus S_H$ such that $uu' \in E(H)$. This implies that $u, u' \in V(G + H) \setminus S$ and $uu' \in E(G + H)$. Thus, $V(G + H) \setminus S$ is a subgraph without isolated vertices, that is, S is a restrained dominating set of $G + H$ Definition 2.3. By Definition 2.5, S is an identifying restrained dominating set of $G + H$.
Case 3. If $S_G \subset V(G)$ and $S_H = V(H)$, then by similar arguments in Case 2, S is an identifying restrained dominating set of $G + H$.
Case 4. If $S_G \subset V(G)$ and $S_H \subset V(H)$, then let $u \in V(G) \setminus S$ and $v \in V(H) \setminus S$. This means that $u, v \in V(G + H) \setminus S$ and $uv \in E(G + H)$. Thus, $V(G + H) \setminus S$ is a subgraph without isolated vertices, that is, S is a restrained dominating set of $G + H$ by Definition 2.3. Since, $S = S_G \cup S_H$ is an identifying code of $G + H$ by hypothesis and by Lemma 2.8, it follows that S is an identifying restrained dominating set of $G + H$ by Definition 2.5.
The following corollary is an immediate consequence of Theorem 2.11.

Corollary 2.12. Let $G = P_m$ and $H = P_n$ with $m = 2k + 1$ for all positive integer k and $n \geq 4$. Then

$$\gamma_r^{ID}(G+H) = \begin{cases} \gamma^{ID}(G) + \gamma^{ID}(H), & if\ n = 2k+3, \\ & or\ n = 2k+2, (n \neq 4)\ and\ (n \neq 6), \\ & or\ n = 4\ and\ m \neq 3, \\ \gamma^{ID}(G) + 3, & if\ n = 6\ and\ m \neq 3. \end{cases}$$

Proof. Let $G = P_m = [v_1, v_2, v_3, \dots, v_m], m = 2k + 1$ for all positive integer k. Then by Lemma 2.9, $S_G = \{v_1, v_3, v_5, \dots, v_m\}$ is a γ^{ID}-set in G. Let $H = P_n = [u_1, u_2, u_3, \dots, u_n], n \geq 4$.
Case 1. Suppose that $n = 2k + 3$ for all positive integer k. By Lemma 2.9, $S_H = \{u_1, u_3, u_5, \dots, u_n\}$ is a γ^{ID}-set in H. This implies that S_G and S_H are identifying codes in G and in H respectively. Since $m = 2k + 1 \geq 3$ for all positive integer k, it follows that $\gamma(\langle S_G \rangle) \neq 1$. If $k = 1$, then for some $S_H \subset V(H)$, $\gamma(\langle S_H \rangle) \neq 1$. If $k > 1$, then for all $S_H \subset V(H)$, $\gamma(\langle S_H \rangle) \neq 1$. Thus, by Theorem 2.11, $S = S_G \cup S_H$ is an identifying restrained dominating set in $G + H$. Hence, $\gamma_r^{ID}(G + H) \leq |S| = |S_G \cup S_H| = |S_G| + |S_H|$. This implies that $\gamma_r^{ID}(G + H) \leq \gamma^{ID}(G) + \gamma^{ID}(H)$. If S is a minimum identifying restrained dominating set in $G + H$, then
$\gamma_r^{ID}(G + H) = |S| = |S_G \cup S_H| = |S_G| + |S_H| = \gamma^{ID}(G) + \gamma^{ID}(H)$.
Suppose that S is not a minimum identifying restrained dominating set in $G + H$. Then, there exists a minimum identifying restrained dominating set $S' = S'_G \cup S'_H$ in $G + H$ such that $|S'| < |S|$. Consider the following cases.
Subcase 1. If $S' = S \setminus \{v\}$ for some $v \in S_G$, then S' is not an identifying code in $G + H$. This is due to the fact that for some $v' \in V(G) \setminus S_G$ such that $vv' \in E(G)$ and for some $v'' \in S_G$ such that $v'v'' \in E(G)$, the $N_{G+H}[v'] \cap S' = S_H \cup \{v''\} = N_{G+H}[v''] \cap S'$. Hence, $N_{G+H}[x] \cap S'$ is not distinct for some $x \in V(G + H)$.
Subcase 2. If $S' = V(G + H) \setminus S$, then $S'_G = V(G) \setminus S_G$ and $S'_H = V(H) \setminus S_H$, that is, $S'_G = \{v_2, v_4, \dots, v_{m-1}\}$ and $S'_H = \{u_2, u_4, \dots, u_{n-1}\}$. This implies that, $N_G[v_1] \cap S' = S'_H \cup \{v_2\} = N_G[v_2] \cap S'$. That is, $N_G[v] \cap S'$ is not distinct for some positive integer $v \in V(G + H)$. Hence, S' is not an identifying code in G.
Subcase 3. If S' is any other identifying restrained dominating set in $G + H$, then consider that $S'_H = \{u_1, u_3, u_5, \dots, u_{n-6}, u_{n-4}, u_{n-3}, u_{n-2}\}$. This implies that

$$S' = S'_G \cup S'_H = \{v_1, v_3, \dots, v_m\} \cup \{u_1, u_2, u_3, \dots, u_{n-6}, u_{n-4}, u_{n-3}, u_{n-2}\}$$

is an identifying restrained dominating set in $G + H$. Since
$|S'_G| = |\{v_1, v_3, \dots, v_m\}| = \frac{m+1}{2}$, and

$$|S'_H| = |\{u_1, u_3, u_5, \dots, u_{n-6}, u_{n-4}, u_{n-3}, u_{n-2}\}| = \frac{(n-6)+1}{2} + 3 = \frac{n+1}{2}.$$

It follows that $|S'| = \frac{m+1}{2} + \frac{n+1}{2} = \frac{m+n+2}{2}$. Note that

$$S = S_G \cup S_H = \{v_1, v_3, v_5, \dots, v_n\} \cup \{u_1, u_3, u_5, \dots, u_m\} = \frac{m+1}{2} + \frac{n+1}{2} = \frac{m+n+2}{2}.$$

Hence, $|S'| = |S|$. This contradict to our assumption that $|S'| < |S|$. Therefore, by subcases 1, 2, and 3, S must be a minimum identifying restrained dominating set in $G + H$, that is,

$$\gamma_r^{ID}(G+H) = |S| = |S_G \cup S_H| = |S_G| + |S_H| = \gamma^{ID}(G) + \gamma^{ID}(H)$$

if $G = P_m$ and $H = P_n$ with $m = 2k+1$ and $n = 2k+3$ for all positive integer k.

Case 2. Suppose that $n = 2k+2,\ n \neq 4$ and $n \neq 6$. By Lemma 2.10, $S_H = \{u_1, u_3, u_5, \dots, u_{n-5}, u_{n-3}, u_{n-2}, u_{n-1}\}$ is a γ^{ID}-set in H. This implies that S_G and S_H are identifying codes in G and in H respectively. Since $m \geq 3$ and $n \geq 8$, it follows that $\gamma(\langle S_G \rangle) \neq 1$ and $\gamma(\langle S_H \rangle) \neq 1$. By Theorem 2.11, $S = S_G \cup S_H$ is an identifying restrained dominating set in $G + H$. Thus,

$$\gamma_r^{ID}(G+H) \leq |S| = |S_G \cup S_H| = |S_G| + |S_H|.$$

This implies that

$$\gamma_r^{ID}(G+H) \leq \gamma^{ID}(G) + \gamma^{ID}(H).$$

If S is a minimum identifying restrained dominating set in $G + H$, then

$$\gamma_r^{ID}(G+H) = |S| = |S_G \cup S_H| = |S_G| + |S_H| = \gamma^{ID}(G) + \gamma^{ID}(H).$$

Suppose that S is not a minimum identifying restrained dominating set in $G + H$. Then, there exists a minimum identifying restrained dominating set $S' = S'_G \cup S'_H$ in $G + H$ such that $|S'| < |S|$. Consider the following subcases.

Subcase 1. If $S' = S \setminus \{v\}$ for some $v \in S_G$, then by similar arguments in the subcase 1 (above), S' is not an identifying code in $G + H$.

Subcase 2. If S' is any other identifying restrained dominating set in $G + H$, then consider that $S'_H = \{u_2, u_3, u_4, \dots, u_{n-4}, u_{n-3}, u_{n-2}\}$. This implies that

$$S' = S'_G \cup S'_H = \{v_1, v_3, \dots, v_m\} \cup \{u_2, u_3, u_4, \dots, u_{n-4}, u_{n-3}, u_{n-2}\}$$

is an identifying restrained dominating set in $G + H$. Since,

$$|S'| = |S'_G \cup S'_H| = |\{v_1, v_3, \dots, v_m\}| \cup |\{u_2, u_3, u_4, \dots, u_{n-4}, u_{n-3}, u_{n-2}\}|$$
$$= \frac{m+1}{2} + \left(1 + \frac{(n-4)}{2} + 2\right) = \frac{m+n+3}{2}, and$$

$$|S| = |S_G \cup S_H| = |\{v_1, v_3, \dots, v_m\}| + |\{u_1, u_3, u_5, \dots, u_{n-5}, u_{n-3}, u_{n-2}, u_{n-1}\}|$$
$$= \frac{m+1}{2} + \left(\frac{(n-3)+1}{2} + 2\right) = \frac{m+n+3}{2}.$$

This implies that $|S'| = \frac{m+n+3}{2} = |S|$ contrary to our assumption that $|S'| < |S|$. Therefore, by subcases 1 and 2, S must be a minimum identifying restrained dominating set in $G + H$, that is,

$$\gamma_r^{ID}(G + H) = |S| = |S_G \cup S_H| = |S_G| + |S_H| = \gamma^{ID}(G) + \gamma^{ID}(H)$$

if $G = P_m$ and $H = P_n$ with $n = 2k + 2$ for all positive integer $k \geq 3$, that is, $n \neq 4$ and $n \neq 6$.

Case 3. Suppose that statement $n = 4$ and $m \neq 3$. By Lemma 2.10, $S_H = \{u_2, u_3, u_4\}$ is a γ^{ID}-set in H. This implies that S_G and S_H are identifying codes in G and in H respectively. Since $n = 4$, it is clear that $S = S_G \cup S_H$ is an identifying restrained dominating set in $G + H$. If S is a minimum identifying restrained dominating set in $G + H$, then

$$\gamma_r^{ID}(G + H) = |S| = |S_G \cup S_H| = |S_G| + |S_H| = \gamma^{ID}(G) + \gamma^{ID}(H).$$

Suppose that S is not a minimum identifying restrained dominating set in $G + H$. Then, there exists a minimum identifying restrained dominating set $S' = S'_G \cup S'_H$ in $G + H$ such that $|S'| < |S|$. Consider the following subcases.

Subcase 1. If $S' = S \setminus \{v\}$ for some $v \in S_G$, then by similar arguments in the subcase 1 (above), S' is not an identifying restrained dominating set in $G + H$.

Subcase 2. If S' is any other identifying restrained dominating set in $G + H$, then consider that $S'_H = \{u_1, u_2, u_3\}$. Then S'_H is a γ^{ID}-set in $H = [u_1, u_2, u_3, u_4]$. Clearly,

$$S' = S'_G \cup S'_H = \{v_1, v_3, \dots, v_m\} \cup \{u_2, u_3, u_4\}$$

is an identifying restrained dominating set in $G + H$. Since

$$\begin{aligned}|S'| &= |S'_G \cup S'_H| = |\{v_1, v_3, \dots, v_m\}| \cup |\{u_1, u_2, u_3\}| \\ &= \frac{m+1}{2} + 3 = \frac{m+7}{2}, \textit{and}\end{aligned}$$

$$\begin{aligned}|S| &= |S_G \cup S_H| = |\{v_1, v_3, \dots, v_m\}| + |\{u_2, u_3, u_4\}| \\ &= \frac{m+1}{2} + 3 = \frac{m+7}{2}.\end{aligned}$$

This implies that $|S'| = \frac{m+7}{2} = |S|$ contrary to our assumption that $|S'| < |S|$. Therefore, by subcases 1 and 2, S must be a minimum identifying restrained dominating set in $G + H$, that is,
$\gamma_r^{ID}(G + H) = |S| = |S_G \cup S_H| = |S_G| + |S_H| = \gamma^{ID}(G) + \gamma^{ID}(H)$.

if $G = P_m$ and $H = P_n$ with $m \neq 3$ and $n = 4$.

Case 4. If $n = 6$ and $m \neq 3$, then consider that $S'_H = \{u_2, u_3, u_4\}$. This implies that

$$S' = S'_G \cup S'_H = \{v_1, v_3, \dots, v_m\} \cup \{u_2, u_3, u_4\}$$

is an identifying restrained dominating set in $G + H$. Since

$$|S'| = |S'_G \cup S'_H| = |\{v_1, v_3, \dots, v_m\}| \cup |\{u_2, u_3, u_4\}|$$
$$= \frac{m+1}{2} + 3 = \frac{m+7}{2}, and$$

$$|S| = |S_G \cup S_H| = |\{v_1, v_3, \dots, v_m\}| + |\{u_1, u_2, u_3\}|$$
$$= \frac{m+1}{2} + 3 = \frac{m+7}{2}.$$

This implies that $|S'| = \frac{m+7}{2} = |S|$ contrary to our assumption that $|S'| < |S|$. Therefore, by S must be a minimum identifying restrained dominating set in $G + H$, that is,

$$\gamma_r^{ID}(G + H) = |S| = |S_G \cup S_H| = |S_G| + |S_H| = \frac{m+1}{2} + 3 = \gamma^{ID}(G) + 3$$

if $G = P_m$ and $H = P_n$ with $m \neq 3$ and $n = 6$.

3. Conclusion and Recommendations

In this work, we introduced a new parameter of domination on graphs - the identifying restrained domination of graphs. The identifying restrained domination in the join of two graphs were characterized. The exact identifying restrained domination number resulting from this binary operation of two graphs were computed. This study will pave a way to new research such as bounds and other binary operations of two graphs. Other parameters involving identifying restrained domination in graphs may also be explored. Finally, the characterization of an identifying restrained domination in graphs may be the subject of further study.

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