Certain Subclasses of Bi-Univalent Functions of σ, ∈ Using Gegenbauer Polynomials

B. R. Nandeesha[1], K. Sathisha[2], M. N. Chandramouleswara[3], M. Sowmya[4]

[1,2,3,4]Senior Scale Lecturer, Department of Science, Government Polytechnic

ABSTRACT
In this article, we introduce a new subclass of analytic functions by using fractional q-differential operator of bi-univalent functions, incorporating Gegenbauer polynomials. For functions within this subclass, we derive upper bounds for the second and third Taylor-Maclaurin coefficients.

Keywords: Analytic function; q-differential operator: bi-univalient function, GEGEX-BAUER POLYNOMIALS

MSC 2020: Primary 30C45; Secondary 30C50

1. INTRODUCTION AND PRELIMINARIES

Let $\Im$ be an analytic function defined on the open unit disk u = {t ∈ C : |t| < 1} satisfying the conditions $\Im(0) = 0$ and $\Im'(0) = 1$. Consequently, $\Im$ can be expressed in the form of the following series expansion:

$$\Im(t) = t + \sum_{n=2}^{\infty} a_n t^n, \quad (t \in u) \qquad (1.1)$$

The class of all functions $\Im$ given by this expansion is denoted by 21, and the class of all functions in 21that are univalent is denoted by S. It is well known that every function $\Im$ in the class S has an inverse map $\Im^{-1}$ given by

$$g(w) = \Im^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots \qquad (1.2)$$

In the context of analytic functions on the open unit disk, a function $\Im \in$ 21 is termed bi-univalent if its inverse map $\Im^{-1}$ is also univalent in u, For more details [9, 4].

For any two analytic functions in the unit disc u, an analytic function $\Im$ is subordinate to an analytic function S, written $\Im(t) \succ S(t)$, if $\Im$ can be expressed as a composition of S and an analytic function w(t), such that $\Im(t) = S(w(t))$, where the function $w(t)$ satisfies the conditions $w(0) = 0$ and $|w(t)| < 1$ for all $t \in U$.

Orthogonal polynomials hold significant importance in various applications spanning mathematics, physics, and engineering. Gegenbauer polynomials, in particular, form a specialized subclass of Jacobi polynomials. For a comprehensive understanding of their fundamental definitions and key properties, refer to [1, 3, 2]

Setting $\alpha > -\frac{1}{2}$, the Gegenbauer polynomials $S_0^{\alpha}(x)$ for n = 2, 3…. are constructed using the following recurrence relation:

$S_0^{\alpha}(x) = 1;$ (1.3)

$S_0^{\alpha}(x) = 2\alpha x;$ (1.4)

$S_0^{\alpha}(x) = 2\alpha x^2 + 2\alpha^2 x^2 - \alpha$ (1.5)

Legendre polynomials and Chebyshev polynomials of the second kind arc specific instances of Gegenbauer polynomials. When α= 1, the Gegenbauer polynomials reduce to the Chebyshev polynomials of the second kind, denoted by $U_n(x) = (S_n^1(x)$ and for $\alpha = \frac{1}{2}$. Similarly, setting $\alpha = \frac{1}{2}$ yields the Legendre polynomials, expressed as $P_n(x) = (S_n^{\frac{1}{2}}(x)$

The generating function for Gegenbauer polynomials is expressed as:

$$S_\alpha(x,t) = \frac{1}{(1-2xt+t^2)^{\alpha}},$$

where $x \in [-1, 1]$. For a fixed value of x, the generating function S_a is analytic in U and can be represented as a Taylor-Maclaurin series

$$S_\alpha(x, t) = \sum_{n=0}^{\infty} S_n^{\alpha}(x) t^n, t \in u \quad (t \in u)$$

1.1. Peliminaries.

Definition 1.1. [5, 6] The. Jackson q-derivative (or q-difference) operator D_q for a function $\Im \in 21$ in a given subset of the set C of complex numbers is defined by

$$(D_q\Im)(t) = \begin{cases} \frac{\Im(qt)-\Im(t)}{(q-1)t} & if\ t \neq 0, \\ \Im'(o) & if\ t = 0, \end{cases} \quad (1.6)$$

Definition 1.2. [8] The fractional q- differintegral operator $\Omega_{q,t}^{\delta}$ for the function $\Im(t)$ of the form (1.1) is defined as

$$\Omega_q^{\delta}, \Im(t) = \alpha_q(2-\delta) t^{\delta} D_q^{\delta}, \Im(t),$$

where $D_{q,t}^{\delta}$ denotes the fractional δ order q-integral of $\Im(t)$ when $-\infty < \delta <$ along with the fractional δ order q-derivative of $\Im(t)$ when $0 \le \delta < 2$.

The expression $\Omega_q^{\delta}, \Im(t)$ is given by:

$$\Omega_q^{\delta}, \Im(t) = t + \sum_{n=2}^{\infty} k_n(q, n, \delta) a_m t^m. \quad (1.7)$$

where

$$k_n(q, n, \delta) = \frac{\alpha_q(n+1)\,\alpha_q(2-\delta)}{\alpha_q(n+1-\delta)}$$

The following Lemma is required to show the main result.

Lemma 1.1 [7] Consider the power series representation $w(t) = \sum_{n-1}^{\infty} 1_n t^n$, subject to the condition $|w(t)| < 1$ for all $t \in U$. The objective is to demonstrate that $|w_1| \le 1$ and $|w_n| \le 1 - |w_1|^2$ for n = 1, 2. . .

2. MAIN RESULTS

Definition 2.1. For a function of the form (1.1) be in the class $\Im_{\Sigma}^{\sigma}(q,x,\sigma,\in,\delta)$ if it satisfies the following conditions:

$$\left(\frac{tD_q\Omega_q^{\delta}\Im(t)}{\Omega_q^{\delta}\Im(t)}\right)^{\sigma}\left(\frac{tD_q\Omega_q^{\delta}\Im(t)}{\Omega_q^{\delta}\Im(z)}\right)^{\in}\succ S_{\alpha}(X,t),$$

and

$$\left(\frac{wD_q\Omega_q^{\delta}h(w)}{\Omega_q^{\delta}h(w)}\right)^{\sigma}\left(\frac{wD_q\Omega_q^{\delta}h(w)}{\Omega_q^{\delta}h(w)}\right)^{\in}\succ S_{\alpha}(X,w),$$

where $0 \le \in 1$, $0 \le \sigma \le 1$, $0 < q < 1$, $0 \le \delta < 2$, $\alpha > -\frac{1}{2}$ w, t $\in$ U and the function g(w) is given by (1.2)

Theorem 2.1. Let $\Im(t)$ be of the form (1.1), then

$$|\alpha_2| \le \frac{2\alpha x\sqrt{2x}}{\sqrt{\left|\left(\left[2q^2\alpha\in(\in-1)+2q^2\alpha\sigma(\sigma-1)-4q\alpha\left[\frac{\in}{1!}\right]+4q^2\alpha\sigma\in-4q\sigma\alpha-2(1+\alpha)\left[q\left[\frac{\in}{1!}+\frac{\sigma}{1!}\right]\right]^2\right]k_2^2 + 4\alpha(1+q+q^2)\left[\frac{\in}{1!}+\frac{\sigma}{1!}\right]K_3\right\}X^2+\left[q\left[\frac{\in}{1!}+\frac{\sigma}{1!}\right]K_2\right]^2\right|}} \tag{2.1}$$

and

$$|\alpha_3| \le \frac{2\alpha x}{(1+q+q^2)\left[\frac{\in}{1!}+\frac{\sigma}{1!}\right]K_3}+\left(\frac{2\alpha X}{Q\left[\frac{\in}{1!}+\frac{\sigma}{1!}\right]K_2}\right)^2 \tag{2.2}$$

where $0 \le \in 1$, $0 \le \sigma \le 1$, $0 < q < 1$, $0 \le \delta < 2$, $\alpha > -\frac{1}{2}$,

$$K_2(q, n, \sigma) = \frac{[2]_q}{[2-\delta]_q},$$

and

$$K_3(q, n, \delta) = \frac{[3]_q\,[2]_q\,[1]_q}{[3-\delta]_q\,[2-\delta]_q},$$

Proof: Suppose $\Im \in \Im_{\Sigma}^{\sigma}$ (q, y, σ, ∈, λ, δ). According to Definition 2.1, this results in the existence of two analytic functions w and v υ in the unit disk U Possessing the following properties w(0) = 0 and v(-) = 0 with w (t) < 1, v(w) < 1 for all w, t ∈U, and

$$\left(\frac{tD_q\Omega_q^{\delta}\Im(t)}{\Omega_q^{\delta}\Im(t)}\right)^{\sigma}\left(\frac{tD_q\Omega_q^{\delta}\Im(t)}{\Omega_q^{\delta}\Im(z)}\right)^{\in}=S_{\alpha}(X,tv(t)), \tag{2.3}$$

and

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$$\left(\frac{wD_q\Omega_q^{\delta}h(w)}{\Omega_q^{\delta}h(w)}\right)^{\sigma}\left(\frac{wD_q\Omega_q^{\delta}h(w)}{\Omega_q^{\delta}h(w)}\right)^{\epsilon}=S_{\alpha}\left(X,tv(w)\right), \quad (2.4)$$

Based on the equalities (2.3) and (2.4) for t, w $\in$U, it follows that

$$\left(\frac{tD_q\Omega_q^{\delta}\Im(t)}{\Omega_q^{\delta}\Im(t)}\right)^{\sigma}\left(\frac{tD_q\Omega_q^{\delta}\Im(t)}{\Omega_q^{\delta}\Im(z)}\right)^{\epsilon}=1+G_1^{\alpha}(x)d_1t+\left[G_1^{\alpha}(x)d_2+G_2^{\alpha}(x)d_1^2\right]t^2+\ldots, \quad (2.5)$$

and

$$\left(\frac{wD_q\Omega_q^{\delta}h(w)}{\Omega_q^{\delta}h(w)}\right)^{\sigma}\left(\frac{wD_q\Omega_q^{\delta}h(w)}{\Omega_q^{\delta}h(w)}\right)^{\epsilon}=1+G_1^{\alpha}(x)E_1w+\left[G_1^{\alpha}(x)E_2+G_2^{\alpha}(x)E_1^2\right]w^2+\ldots \quad (2.6)$$

where $tv(z) = \sum_{j=1}^{\infty} d^j z^j$ and $v(w) = \sum_{j=1}^{\infty} E_j w^j$ (2.7)

Using Lemma 1.1, we achieved

$|d_j| \leq 1$ and $|E_j| \leq 1 \ \forall \ j \in N$ (2.8)

Now, comparing the coefficients in equation (2.5) and equation (2.6), we have

$$\left[\frac{\in}{1!}+\frac{\sigma}{1!}\right]qa_2k_2=G_1^{\sigma}(x)d_1, \quad (2.9)$$

$$q\left[q\frac{\in(\in-1)}{2!}-\frac{\in}{1!}+q\frac{\sigma(\sigma-1)}{2!}+q\frac{\sigma\in}{1!}+\frac{\sigma}{1!}\right]a_2^2k_2^2+(1+q+q^2)\left[\frac{\in}{1!}+\frac{\sigma}{1!}\right]a_3k_3=G_1^{\alpha}(x)d_2+G_2^{\sigma}(x)d_1^2,$$

$$-\left[\frac{\in}{1!}+\frac{\sigma}{1!}\right]qa_2k_2=G_1^{\alpha}(x)E_1, \quad (2.11)$$

$$\left\{\left[\frac{\in(\in-1)}{2!}q^2-\frac{\in}{1!}+q+\frac{\sigma\in}{1!}q^2+\frac{\sigma(\sigma-1)}{2!}q^2-q\frac{\sigma}{1!}\right]k_2^2+2(1+q+q^2)\left[\frac{\in}{1!}+\frac{\sigma}{1!}\right]k_3\right\}a_2^2$$

$$-(1+q+q^2)\left[\frac{\in}{1!}+\frac{\sigma}{1!}\right]a_3k_3=G_1^{\alpha}(x)E_2+G_2^{\sigma}(x)E_1^2. \quad (2.12)$$

Using (2.9) and (2.11), we have

$$D_1 = - E_1, \quad (2.13)$$

with

$$2q^2\left[\frac{\in}{1!}+\frac{\sigma}{1!}\right]^2 a_2^2k_2^2=[G_1^{\alpha}(x)]^2\ [d_1^2+E_1^2]. \quad (2.14)$$

Adding (2.10) and (2.11), we get

Using (2.14) and substituting the value of $(d_1^2+E_1^2)$ in the right side of (2.15), we achieved

$$2\left\{\left[\frac{\in(\in-1)}{2!}q^2[G_1^{\alpha}(x)]^2-\frac{\in}{1!}q[G_1^{\alpha}(x)]^2\right.\right.$$

$$+\frac{\in}{1!}q^2\,[G_1^\alpha(x)]^2+\frac{\sigma(\sigma-1)}{2!}q^2[G_1^\alpha(x)]^2-\frac{\sigma}{1!}q[G_1^\alpha(x)]^2$$

$$-q^2\,[\frac{\in}{1!}+\frac{\sigma}{1!}]^2\,G_2^\alpha(x)\Big]k_2^2+(1+q+q^2)\,[\frac{\in}{1!}+\frac{\sigma}{1!}][G_1^\alpha(x)]^2k_3\Big\}\frac{1}{[G_1^\alpha(x)]^2}a_2^2$$

$$=[G_1^\alpha(x)](E_2+d_2). \qquad (2.16)$$

By applying (1.4), (1.5), (2.8), and (2.16), we conclude that (2.1) satisfies. Now, subtract (2.12) from (2.10), we have

$$2(1+q+q^2)\,[\frac{\in}{1!}+\frac{\sigma}{1!}]k_3\left(a_3-a_2^2\right)=G_1^\alpha(x)(d_2-E_2)+G_2^\alpha(x)\left(d_1^2-E_1^2\right). \qquad (2.17)$$

Using (22) with (23) in (26), we get

$$a_3=\frac{[G_1^\alpha(x)]^2(d_1^2+E_1^2)}{2q^2\left(\frac{\in}{1!}+\frac{\sigma}{1!}\right)^2k_2^2}+\frac{[G_1^\alpha(x)](d_2-E_2)}{2(1+q+q^2)\left(\frac{\in}{1!}+\frac{\sigma}{1!}\right)k_3} \qquad (2.18)$$

Applying (1.4) in (2.18), We readily achieve

$$|a_3|=\frac{2\sigma x}{(1+q+q^2)\left(\frac{\in}{1!}+\frac{\sigma}{1!}\right)k_3}+\frac{(2\sigma x)^2}{q^2\left(\frac{\in}{1!}+\frac{\sigma}{1!}\right)^2k_2^2} \qquad (2.19)$$

This Completes the proof of the Theorem.

By applying equations (2.1) and (2.2) with the parameter α = 1, we obtain the following notworthy corollary. The initial coefficient estimates are associated with the Chebyshev polynomials of the second kind. Since the proof follows a methodology similar to that of the preceding theorems, the detailed steps are omitted for conciseness.

Corollary 2.1. Let the function ℑ, as defined by equation (1.1), belong to the class $\Im_\Sigma(q, x, \sigma, \in, \delta)$

$$|a_2|\le\frac{2x\sqrt{2x}}{\sqrt{\left\{\left[2\in(\in-1)q^2+2\sigma(\sigma-1)q^2-4\frac{\in}{1!}q+4\sigma\in q^2-4\sigma q-4\left[q[\frac{\in}{1!}+\frac{\sigma}{1!}]\right]^2\right]k_2^2+4(1+q+q^2)\left[\frac{\in}{1!}+\frac{\sigma}{1!}\right]k_3\right\}x^2+\left[q[\frac{\in}{1!}+\frac{\sigma}{1!}]k_2\right]^2}}$$

with

$$|a_3|\le\frac{2x}{(1+q+q^2)\frac{\in}{1!}+\frac{\sigma}{1!}k_3}+\left(\frac{2x}{q\left[\frac{\in}{1!}+\frac{\sigma}{1!}\right]k_2}\right)^2$$

where $0\le\in\le1$, $0\le\sigma\le1$, $0<q<1$, $0\le\delta<2$,

$$k_2(q,n,\delta)=\frac{[2]_q}{[2-\delta]_q},$$

and

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$$k_3(q, n, \delta) = \frac{[3]_q[2]_q[1]_q}{[3-\delta]_q[2-\delta]_q},$$

By setting $\alpha = \frac{1}{2}$, an alternative corollary for Legendre polynomials can be derived, providing further insight into their properties and applications.

Corollary 2.2. Let the function $\Im$, defined by equation (1.1), belong to the class $\Im_{\Sigma}^{\frac{1}{2}}$ (q, x, σ, ∈, δ).

$$|a_2| \leq \frac{2x\sqrt{2x}}{\sqrt{\left|\left\{\left[\in(\in-1)q^2+\sigma(\sigma-1)q^2-2\frac{\in}{1!}q+2\sigma\in q^2-2\sigma q-3\left[q[\frac{\in}{1!}+\frac{\sigma}{1!}]\right]^2\right]k_2^2 +2(1+q+q^2)\left[\frac{\in}{1!}+\frac{\sigma}{1!}\right]k_3\right\}x^2+\left[q[\frac{\in}{1!}+\frac{\sigma}{1!}]k_2\right]^2\right|}}$$

and

$$|a_3| \leq \frac{x}{(1+q+q^2)\left[\frac{\in}{1!}+\frac{\sigma}{1!}\right]k_3}+\left(\frac{x}{q\left[\frac{\in}{1!}+\frac{\sigma}{1!}\right]k_2}\right)^2$$

where $0 \leq \in \leq 1$, $0 \leq \sigma \leq 1$, $0 < q < 1$, $0 \leq \delta < 2$,

$$k_2(q, n, \delta) = \frac{[2]_q}{[2-\delta]_q},$$

and

$$k_3(q, n, \delta) = \frac{[3]_q[2]_q[1]_q}{[3-\delta]_q[2-\delta]_q},$$

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