International Journal for Multidisciplinary Research (IJFMR)
E-ISSN: 2582-2160 • Website: www.ijfmr.com • Email: editor@ijfmr.com

# On Hamming Distances of $(\alpha + u\beta)$-Constacyclic Codes of Length $4p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$

**Madhu Kant Thakur**

Researcher

**Abstract**
In this paper, we discuss all the Hamming distances of $(\alpha + u\beta)$-constacyclic codes of $4p^s$ length over the finite chain ring $R = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ with $u^2 = 0$. Using the structures of $(\alpha + u\beta)$-constacyclic codes of length $4p^s$ over $R$, the Hamming distances of all constacyclic codes of length $4p^s$ over $R$ are determined. We provide some parameters of good codes over $R$ as examples, from which some are new in literature.

**Keywords.** Repeated-Root Codes, Constacyclic Codes, Hamming Distance, Finite Chain Ring, MDS Codes.

## 1 Introduction

The constacyclic codes make a most important class of linear codes in theories of error-correcting codes because they generalize the structural properties of cyclic codes, which are the most studied family of linear codes till now due to their rich algebraic structures. These codes also have practical fulfillment due to their encoding simply with shift registers. Nowadays, there has been a great curiosity in the study of the constacyclic codes for the coding theorists.

Let $\lambda$ be a unit in a finite commutative ring $R = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}; u^2 = 0$. Then $\lambda$-constacyclic codes of length $4p^s$ are ideals of the ambient ring $R_\lambda = \frac{R[x]}{\langle x^{4p^s} - \lambda \rangle}$. Constacyclic codes are said to be simple-root constacyclic codes if $\gcd(n, p) = 1$ and repeated-root constacyclic codes if $\gcd(n, p) = p$.

The repeated-root constacyclic codes were first introduced by Castagnoli et al. [18] and van Lint [19] in 1991. Although these codes are asymptotically bad but there exist some optimal codes which motivate researchers to work further on these codes. Later, lot of researchers studied the repeated-root constacyclic codes over finite fields [13, 14, 15, 16, 17, 18, 19] and finite chain rings [1, 2, 8, 9, 10, 11, 12].

The structures of constacyclic codes of $p^s, 2p^s, 3p^s, 4p^s$ over $\mathbb{F}_{p^m}$ were determined by Dinh in a series of papers [12, 13, 14, 15]. In [12], Dinh obtained all the Hamming distances of cyclic codes of length $p^s$ over $\mathbb{F}_{p^m}$. Later, in [16], Özadam et al. computed the Hamming distances of all cyclic codes of $2p^s$ length . The Hamming distances of cyclic codes of length $3p^s$ were determined in [17]. In [3], based on the weight-retaining property of polynomials, López-Permouth et al. determined the Hamming distances of some classes of constacyclic codes of length $np^s$ over the finite field $\mathbb{F}_{p^m}$. Later, in [5], Dinh et al. computed all the Hamming distances of the constacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m}$.

The class of finite commutative chain rings of the form $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ has been widely used as alphabets of specific constacyclic codes.

In a series of studies [1, 8, 9, 10, 11, 12], Dinh et al. established the structures of some classes of constacyclic codes of certain lengths over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$. In [2], Guenda et al. determined the algebraic structures of constacyclic codes of length $wp^s$ in generalized form over the finite ring $\frac{\mathbb{F}_{p^m}[u]}{\langle u^e \rangle}$. However, a minimal amount of work has been done on the computation of the Hamming distances of constacyclic codes due to computational complexity. In [1], Dinh determined all the Hamming distances of $(\alpha + u\beta)$-constacyclic codes of length $p^s$ over $R$ using the results of [3, 15]. Later, in [4], Dinh et al. determined the Hamming distances of all $\gamma$-constacyclic codes of prime power length over $R$.

The rest of this paper is organized as follows. Section 2 contains the structures of all $(\alpha + u\beta)$-constacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ and some other preliminary results. In Section 3, we compute all the Hamming distances of $(\alpha + u\beta)$-constacyclic codes of length $4p^s$ over $R$. In section 4, we provide some examples for different units of $R$. In section 5, we explore all maximum distance separable $(\alpha + u\beta)$-constacyclic codes of length $4p^s$ over $R$ and we conclude the paper in section 6.

## 2 Preliminaries

Let $R = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}; u^2 = 0$ be a finite commutative ring, where $p$ is an odd prime. The ring $R$ can also be expressed as $R = \frac{\mathbb{F}_{p^m}[u]}{\langle u^2 \rangle}$, where $p$ is prime. The elements of $R$ can be represented as $\{a + ub \mid a, b \in \mathbb{F}_{p^m}, u^2 = 0\}$. The ring $R$ has $p^m(p^m - 1)$ units, which are of two types $\lambda = (\alpha + u\beta)$ and $\lambda = \gamma$; where $\alpha, \beta, \gamma \in \mathbb{F}_{p^m}$. A code $C$ of length $n$ over $R$ is a non-empty subset of $R^n$ and $C$ is linear over $R$ if it is an $R$-submodule of $R^n$. Let $\lambda$ be a unit in $R$ and $\tau_\lambda$ be a map from $R^n$ to $R^n$ given by

$$\tau_\lambda(c_0, c_1, \ldots, c_{n-1}) = (\lambda c_{n-1}, c_0, \ldots, c_{n-2})$$

A linear code $C$ over $R$ is $\lambda$-constacyclic if $\tau_\lambda(C) = C$. The code $C$ is called cyclic and negacyclic code over $R$ according as $\lambda = 1$ and $\lambda = -1$ respectively.

The codeword $e = (e_0, e_1, \ldots, e_{n-1}) \in C$ can be expressed as the polynomial $e(x) = e_0 + e_1 x + \ldots + e_{n-1}x^{n-1}$ of $\frac{R[x]}{\langle x^n - \lambda \rangle}$.

Let $C$ be a code of length $n$ over $R$ and a codeword $e = (e_0, e_1, \ldots, e_{n-1}) \in R^n$. Then, the Hamming weight $wt_H(e)$ of a codeword $e$ is the number of nonzero components i.e.

$$wt_H(e) = \sum_{0}^{n-1} wt_H(e_i) \; ; \text{ where } wt_H(e_i) = 1 \text{ if } e_i \neq 0 \text{ and } wt_H(e_i) = 0 \text{ if } e_i = 0.$$

The minimum weight of a code $C$ is denoted by $wt_H(C)$ and is the smallest weight among all its nonzero codewords. The Hamming distance of $C$ is defined by $d_H(C) = \min\{wt_H(e) | e \neq 0, e \in C\}$.

The following proposition is one of the important result for the constacyclic codes.

**Proposition 2.1** [7] A linear code $C$ of length $n$ over $R$ is $\lambda$-constacyclic code over $R$ if and only if $C$ is an ideal of $\frac{R[x]}{\langle x^n - \lambda \rangle}$.

In [2], Guenda et al. discussed the structure of repeated-root constacyclic codes of general length $wp^s$ over $\mathbb{F}_p^m + u\mathbb{F}_p^m + \cdots + u^{e-1}\mathbb{F}_p^m$ with $u^e = 0$. Here, we present those structure for $(\alpha + u\beta)$-constacyclic codes of length $4p^s$ over $R$.

**Theorem 2.2** [2] Let $gcd(w,p)=1$ and $\lambda=(\alpha+u\beta)$ such that $\alpha={\alpha_1}^{p^s}\in\mathbb{F}_{p^m}^*$ and $\beta\neq 0$. Then, $x^{4p^s}-\alpha=\prod_{l\in I}{f_l}^{p^s}$ factors uniquely as the product of irreducible polynomials in $\mathbb{F}_{p^m}$ and $R$ is a principle ideal ring whose ideals are generated by

$$\langle\prod_{l\in J\subset I}{f_l}^{s_i}\rangle$$

where $0\leq s_i\leq 2p^s$.

From the above Theorem, structures of all $(\alpha+u\beta)$-constacyclic codes of length $4p^s$ over $R$ are summarized as following in Table I:

**Table I: Strucures of $(\alpha+u\beta)$-constacyclic codes of length $4p^s$**

| Structure | p,m | Type of $\lambda$ | Generator of $(\alpha+u\beta)$-constacyclic codes |
|---|---|---|---|
| I | $p^m\equiv 3(\mathrm{mod}4)$ | $\lambda$ is a square | $\langle(x-\alpha_0)^i(x+\alpha_0)^j(x^2+{\alpha_0}^2)^k\rangle$ |
| II | $p^m\equiv 3(\mathrm{mod}4)$ | $\lambda$ is non-square | $\langle(x^2+2\delta x+2\delta^2)^i(x^2-2\delta x+2\delta^2)^j\rangle$ |
| III | $p^m\equiv 1(\mathrm{mod}4)$ | $\lambda$ is a square of the form $\lambda=\lambda_0^4$ | $\langle(x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle$ |
| IV | $p^m\equiv 1(\mathrm{mod}4)$ | $\lambda$ is a square of the form $\lambda=\lambda_0^2$ such that $\lambda_0$ is non-square. | $\langle(x^2-\alpha_0)^i(x^2+\alpha_0)^j\rangle$ |
| V | $p^m\equiv 1(\mathrm{mod}4)$ | $\lambda$ is a non-square. | $\langle(x^4-\alpha_0)^i\rangle$ |

**Theorem 2.3** Let $C$ be a $(\alpha+u\beta)$-constacyclic code of length $4p^s$ over $R$, then the number of codewords of C, is determined as follows.

- If $C=\langle(x-\alpha_0)^i(x+\alpha_0)^j(x^2+{\alpha_0}^2)^k\rangle$, then $|C|=p^{m(8p^s-i-j-2k)}$.
- If $C=\langle(x^2+2\delta x+2\delta^2)^i(x^2-2\delta x+2\delta^2)^j\rangle$, then $|C|=p^{m(8p^s-2i-2j)}$.
- If $C=\langle(x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle$, then $|C|=p^{m(8p^s-i-j-k-l)}$.
- If $C=\langle(x^2-\alpha_0)^i(x^2+\alpha_0)^j\rangle$, then $|C|=p^{m(8p^s-2i-2j)}$.
- If $C=\langle(x^4-\alpha_0)^i\rangle$, then $|C|=p^{m(8p^s-4i)}$.

The Hamming distances of $\lambda$-constacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m}$ have been given in [3, 5]. We are listing those results here as follows-

**Theorem 2.4 [5]** Let $C=\langle(x-\lambda_0)^i(x+\lambda_0)^j(x^2+{\lambda_0}^2)^k\rangle\subseteq\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\lambda\rangle}$ for $0\leq k\leq j\leq i\leq p^s$, and $d_H(C)$ is determined by

- $d_H(C)=1, \text{if} i=j=k=0$
- $d_H(C)=2, \text{if } k=0 \ and \ 0\leq j\leq i\leq p^s(\text{butnot } i=j=0)$
- $d_H(C)=\min\{(\rho_0+1)p^{\kappa_0},2(\rho_2+1)p^{\kappa_2}\}, \text{if } p^s-p^{s-\kappa_0}+(\rho_0-1)p^{s-\kappa_0-1}+1\leq i \leq p^s-p^{s-\kappa_0}+\rho_0p^{s-\kappa_0-1}, p^s-p^{s-\kappa_1}+(\rho_1-1)p^{s-\kappa_1-1}+1\leq j \leq p^s-p^{s-\kappa_1}+\rho_1p^{s-\kappa_1-1}, p^s-p^{s-\kappa_2}+(\rho_2-1)p^{s-\kappa_2-1}+1\leq k \leq p^s-p^{s-\kappa_2}+\rho_2p^{s-\kappa_2-1}$

- $d_H(C) = 2(\rho_2 + 1)p^{\kappa_2}$, if $i = p^s, p^s - p^{s-\kappa_1} + (\rho_1 - 1)p^{s-\kappa_1-1} + 1 \leq j \leq p^s, p^s - p^{s-\kappa_2} + (\rho_2 - 1)p^{s-\kappa_2-1} + 1 \leq k \leq p^s - p^{s-\kappa_2} + \rho_2 p^{s-\kappa_2-1}$
- $d_H(C) = 0$, if $i = j = k = p^s$

where $1 \leq \rho_0, \rho_1, \rho_2 \leq p - 1,\ 0 \leq \kappa_2 \leq \kappa_1 \leq \kappa_0 \leq s - 1$.

**Remark 2.5** [5] It is easy to check that the corresponding case with $k \leq i \leq j$ has the same Hamming distances as Theorem 2.4 by symmetry.

**Theorem 2.6** [5] Let $C = \langle (x - \lambda_0)^i (x + \lambda_0)^j (x^2 + {\lambda_0}^2)^k \rangle \subseteq \frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s} - \lambda \rangle}$ for $0 \leq j \leq k \leq i \leq p^s$, and $d_H(C)$ is determined by

- $d_H(C) = 1$, if $i = j = k = 0$
- $d_H(C) = 2$, if $j = k = 0\ and\ 1 \leq i \leq p^s$ or $j = 0\ and\ 1 \leq k \leq i \leq p^{s-1}$
- $d_H(C) = 3$, if $j = 0,\quad 1 \leq k \leq 2p^{s-1}\ and\ p^{s-1} + 1 \leq i \leq 2p^{s-1}$
- $d_H(C) = 4$, if $j = 0\ and\ 1 \leq k \leq p^s\ and\ 2p^{s-1} + 1 \leq i \leq p^s$
- $d_H(C) = \min\{(\rho_0 + 1)p^{\kappa_0}, 2(\rho_1 + 1)p^{\kappa_1}, 4(\rho_2 + 1)p^{\kappa_2}\}$, if $p^s - p^{s-\kappa_0} + (\rho_0 - 1)p^{s-\kappa_0-1} + 1 \leq i \leq p^s - p^{s-\kappa_0} + \rho_0 p^{s-\kappa_0-1}, p^s - p^{s-\kappa_1} + (\rho_1 - 1)p^{s-\kappa_1-1} + 1 \leq k \leq p^s - p^{s-\kappa_1} + \rho_1 p^{s-\kappa_1-1}, p^s - p^{s-\kappa_2} + (\rho_2 - 1)p^{s-\kappa_2-1} + 1 \leq j \leq p^s - p^{s-\kappa_2} + \rho_2 p^{s-\kappa_2-1}$
- $d_H(C) = \min\{(2(\rho_1 + 1)p^{\kappa_1}, 4(\rho_2 + 1)p^{\kappa_2}\}$, if $i = p^s, p^s - p^{s-\kappa_1} + (\rho_1 - 1)p^{s-\kappa_1-1} + 1 \leq k \leq p^s - p^{s-\kappa_1} + \rho_1 p^{s-\kappa_1-1}, p^s - p^{s-\kappa_2} + (\rho_2 - 1)p^{s-\kappa_2-1} + 1 \leq j \leq p^s - p^{s-\kappa_2} + \rho_2 p^{s-\kappa_2-1}$
- $d_H(C) = 4(\rho_2 + 1)p^{\kappa_2}$, if $i = k = p^s, p^s - p^{s-\kappa_2} + (\rho_2 - 1)p^{s-\kappa_2-1} + 1 \leq j \leq p^s - p^{s-\kappa_2} + \rho_2 p^{s-\kappa_2-1}$
- $d_H(C) = 0$, if $i = j = k = p^s$

where $1 \leq \rho_0, \rho_1, \rho_2 \leq p - 1,\ 0 \leq \kappa_2 \leq \kappa_1 \leq \kappa_0 \leq s - 1$.

**Remark 2.7** [5] It is easy to check that the corresponding cases with $i \leq j \leq k, i \leq k \leq j$ and $j \leq i \leq k$ have the same Hamming distances as case $j \leq k \leq i$ in the above Theorem 2.6. **Theorem 2.8** [5] Let $C = \langle (x^2 + 2\delta x + 2\delta^2)^i (x^2 - 2\delta x + 2\delta^2)^j \rangle \subseteq \frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s} - \lambda \rangle}$ for $0 \leq j \leq i \leq p^s$ and $d_H(C)$ is determined by

- $d_H(C) = 1$, if $i = j = 0$
- $d_H(C) = 2$, if $j = 0\ and\ 1 \leq i \leq p^{s-1}$
- $d_H(C) = 3$, If $j = 0,\quad and\ p^{s-1} + 1 \leq i \leq p^s$
- $d_H(C) = \min\{(\rho_0 + 1)p^{\kappa_0}, 3(\rho_1 + 1)p^{\kappa_1}\}$, if $p^s - p^{s-\kappa_0} + (\rho_0 - 1)p^{s-\kappa_0-1} + 1 \leq i \leq p^s - p^{s-\kappa_0} + \rho_0 p^{s-\kappa_0-1}, p^s - p^{s-\kappa_1} + (\rho_1 - 1)p^{s-\kappa_1-1} + 1 \leq j \leq p^s - p^{s-\kappa_1} + \rho_1 p^{s-\kappa_1-1}$
- $d_H(C) = 3(\rho_1 + 1)p^{\kappa_1}$, if $i = p^s, p^s - p^{s-\kappa_1} + (\rho_1 - 1)p^{s-\kappa_1-1} + 1 \leq j \leq p^s - p^{s-\kappa_1} + \rho_1 p^{s-\kappa_1-1}$
- $d_H(C) = 0$, if $i = j = p^s$

where $1 \leq \rho_0, \rho_1 \leq p - 1,\ 0 \leq \kappa_1 \leq \kappa_0 \leq s - 1$.

**Remark 2.9** [5] It is easy to check that the corresponding case with $i \leq j$ has the same Hamming distances as case $j \leq i$ in the above Theorem 2.8.

**Theorem 2.10** [5] Let $C = \langle (x - \alpha_0)^i (x + \alpha_0)^j (x - \eta\alpha_0)^k (x + \eta\alpha_0)^l \rangle \subseteq \frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s} - \lambda \rangle}$ for $0 \leq l \leq k \leq$

$j \leq i \leq p^s$, and $d_H(C)$ is determined by

- $d_H(C) = 1$, if $i = j = k = l = 0$
- $d_H(C) = 2$, if $k = l = 0$ *and* $0 \leq j \leq i \leq p^s$ $(but not\, i = j = 0)$ or $l = 0$ *and* $1 \leq k \leq j \leq i \leq p^{s-1}$
- $d_H(C) = 4$, if $l = 0$, $1 \leq k \leq j \leq p^s$ *and* $p^{s-1} + 1 \leq i \leq p^s$
- $d_H(C) = \min\{(\rho_0 + 1)p^{\kappa_0}, 2(\rho_2 + 1)p^{\kappa_2}, 4(\rho_3 + 1)p^{\kappa_3}\}$, if $p^s - p^{s-\kappa_0} + (\rho_0 - 1)p^{s-\kappa_0-1} + 1 \leq i \leq p^s - p^{s-\kappa_0} + \rho_0 p^{s-\kappa_0-1}, p^s - p^{s-\kappa_1} + (\rho_1 - 1)p^{s-\kappa_1-1} + 1 \leq j \leq p^s - p^{s-\kappa_1} + \rho_1 p^{s-\kappa_1-1}, p^s - p^{s-\kappa_2} + (\rho_2 - 1)p^{s-\kappa_2-1} + 1 \leq k \leq p^s - p^{s-\kappa_2} + \rho_2 p^{s-\kappa_2-1}, p^s - p^{s-\kappa_3} + (\rho_3 - 1)p^{s-\kappa_3-1} + 1 \leq l \leq p^s - p^{s-\kappa_3} + \rho_3 p^{s-\kappa_3-1}$
  - $d_H(C) = \min\{2(\rho_2 + 1)p^{\kappa_2}, 4(\rho_3 + 1)p^{\kappa_3}\}$ if $i = p^s, p^s - p^{s-\kappa_1} + (\rho_1 - 1)p^{s-\kappa_1-1} + 1 \leq j \leq p^s, p^s - p^{s-\kappa_2} + (\rho_2 - 1)p^{s-\kappa_2-1} + 1 \leq k \leq p^s - p^{s-\kappa_2} + \rho_2 p^{s-\kappa_2-1}, p^s - p^{s-\kappa_3} + (\rho_3 - 1)p^{s-\kappa_3-1} + 1 \leq l \leq p^s - p^{s-\kappa_3} + \rho_3 p^{s-\kappa_3-1}$
- $d_H(C) = 4(\rho_3 + 1)p^{\kappa_3}$ if $i = j = k = p^s, p^s - p^{s-\kappa_3} + (\rho_3 - 1)p^{s-\kappa_3-1} + 1 \leq l \leq p^s - p^{s-\kappa_3} + \rho_3 p^{s-\kappa_3-1}$
- $d_H(C) = 0$, if $i = j = k = l = p^s$

where $1 \leq \rho_0, \rho_1, \rho_2, \rho_3 \leq p - 1$, $0 \leq \kappa_3 \leq \kappa_2 \leq \kappa_1 \leq \kappa_0 \leq s - 1$.

**Remark 2.11** [5] It is easy to check that the corresponding cases with $k \leq l \leq j \leq i$, $k \leq l \leq i \leq j$, $l \leq k \leq i \leq j$, $j \leq i \leq l \leq k$, $i \leq j \leq k \leq l$, $i \leq j \leq l \leq k$ and $j \leq i \leq k \leq l$ have the same Hamming distances as $l \leq k \leq j \leq i$ in the above Theorem.

**Theorem 2.12** [5] Let $C = \langle (x - \alpha_0)^i (x + \alpha_0)^j (x - \eta\alpha_0)^k (x + \eta\alpha_0)^l \rangle \subseteq \frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s} - \lambda \rangle}$ for $0 \leq l \leq j \leq k \leq i \leq p^s$, and $d_H(C)$ is determined by

- $d_H(C) = 1$, if $i = j = k = l = 0$
- $d_H(C) = 2$, if $j = k = l = 0$ *and* $1 \leq i \leq p^s$ or $l = 0$ *and* $1 \leq j \leq k \leq i \leq p^{s-1}$
  - $d_H(C) = 3$, if $j = l = 0$, $1 \leq k \leq p^s$ *and* $p^{s-1} + 1 \leq i \leq p^s$ or $l = 0$, $1 \leq j \leq k \leq 2p^{s-1}$ *and* $p^{s-1} + 1 \leq i \leq 2p^{s-1}$
- $d_H(C) = 4$, if $l = 0$, $1 \leq j \leq k \leq p^s$ *and* $2p^{s-1} + 1 \leq i \leq p^s$
  - $d_H(C) = \min\{(\rho_0 + 1)p^{\kappa_0}, 2(\rho_1 + 1)p^{\kappa_1}, 3(\rho_2 + 1)p^{\kappa_2}, 4(\rho_3 + 1)p^{\kappa_3}\}$, if $p^s - p^{s-\kappa_0} + (\rho_0 - 1)p^{s-\kappa_0-1} + 1 \leq i \leq p^s - p^{s-\kappa_0} + \rho_0 p^{s-\kappa_0-1}, p^s - p^{s-\kappa_1} + (\rho_1 - 1)p^{s-\kappa_1-1} + 1 \leq j \leq p^s - p^{s-\kappa_1} + \rho_1 p^{s-\kappa_1-1}, p^s - p^{s-\kappa_2} + (\rho_2 - 1)p^{s-\kappa_2-1} + 1 \leq k \leq p^s - p^{s-\kappa_2} + \rho_2 p^{s-\kappa_2-1}, p^s - p^{s-\kappa_3} + (\rho_3 - 1)p^{s-\kappa_3-1} + 1 \leq l \leq p^s - p^{s-\kappa_3} + \rho_3 p^{s-\kappa_3-1}$
  - $d_H(C) = \min\{2(\rho_1 + 1)p^{\kappa_1}, 3(\rho_2 + 1)p^{\kappa_2}, 4(\rho_3 + 1)p^{\kappa_3}\}$, if $i = p^s, p^s - p^{s-\kappa_1} + (\rho_1 - 1)p^{s-\kappa_1-1} + 1 \leq j \leq p^s - p^{s-\kappa_1} + \rho_1 p^{s-\kappa_1-1}, p^s - p^{s-\kappa_2} + (\rho_2 - 1)p^{s-\kappa_2-1} + 1 \leq k \leq p^s - p^{s-\kappa_2} + \rho_2 p^{s-\kappa_2-1}, p^s - p^{s-\kappa_3} + (\rho_3 - 1)p^{s-\kappa_3-1} + 1 \leq l \leq p^s - p^{s-\kappa_3} + \rho_3 p^{s-\kappa_3-1}$
- $d_H(C) = \min\{3(\rho_2 + 1)p^{\kappa_2}, 4(\rho_3 + 1)p^{\kappa_3}\}$, if $i = j = p^s, p^s - p^{s-\kappa_2} + (\rho_2 - 1)p^{s-\kappa_2-1} + 1 \leq k \leq p^s - p^{s-\kappa_2} + \rho_2 p^{s-\kappa_2-1}, p^s - p^{s-\kappa_3} + (\rho_3 - 1)p^{s-\kappa_3-1} + 1 \leq l \leq p^s - p^{s-\kappa_3} + \rho_3 p^{s-\kappa_3-1}$

- $d_H(C)=4(\rho_3+1)p^{\kappa_3}$, if $i=j=k=p^s, p^s-p^{s-\kappa_3}+(\rho_3-1)p^{s-\kappa_3-1}+1\le l \le p^s-p^{s-\kappa_3}+\rho_3 p^{s-\kappa_3-1}$
- $d_H(C)=0$, if $i=j=k=l=p^s$

where $1\le \rho_0,\rho_1,\rho_2,\rho_3\le p-1,\ 0\le \kappa_3\le\kappa_2\le\kappa_1\le\kappa_0\le s-1$.

**Remark 2.13** [5] It is easy to check that the corresponding cases with $j\le l\le k\le i, j\le k\le l\le i, k\le j\le l\le i, l\le i\le k\le j, i\le k\le l\le j, k\le i\le l\le j, i\le l\le k\le j, l\le i\le j\le k, l\le j\le i\le k, i\le j\le l\le k, j\le l\le i\le k, k\le j\le i\le l, i\le k\le j\le l, k\le i\le j\le l$ and $j\le k\le i\le l$ have the same Hamming distances as case $l\le j\le k\le i$ in the above Theorem.

**Theorem 2.14** [3] Let $C=\langle (x^2-{\alpha_0}^2)^i(x^2+{\alpha_0}^2)^j\rangle \subseteq \frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\lambda\rangle}$ for $0\le j\le i\le p^s$ and $d_H(C)$ is determined by

- $d_H(C)=1$, if$i=j=0$
- $d_H(C)=2$, if $j=0$ $and$ $1\le i\le p^s$
- $d_H(C)=\min\{(\rho_0+1)p^{\kappa_0}, 2(\rho_1+1)p^{\kappa_1}\}$, if $p^s-p^{s-\kappa_0}+(\rho_0-1)p^{s-\kappa_0-1}+1\le i \le p^s-p^{s-\kappa_0}+\rho_0 p^{s-\kappa_0-1}, p^s-p^{s-\kappa_1}+(\rho_1-1)p^{s-\kappa_1-1}+1\le j \le p^s-p^{s-\kappa_1}+\rho_1 p^{s-\kappa_1-1}$
- $d_H(C)=2(\rho_1+1)p^{\kappa_1}$, if $i=p^s, p^s-p^{s-\kappa_1}+(\rho_1-1)p^{s-\kappa_1-1}+1\le j \le p^s-p^{s-\kappa_1}+\rho_1 p^{s-\kappa_1-1}$
- $d_H(C)=0$, if $i=j=p^s$

where $1\le\rho_0,\rho_1\le p-1,\ 0\le\kappa_1\le\kappa_0\le s-1$.

**Remark 2.15** [3] It is easy to check that the corresponding case with $i\le j$ has the same Hamming distances as case $j\le i$ in the above Theorem.

**Theorem 2.16** [12] Let $C=\langle (x^4-\alpha_0)^i\rangle \subseteq \frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha\rangle}$, for $i\in\{0,1,\dots,p^s\}$, then the Hamming distance $d_H(C)$ is completely determined by

- $d_H(C)=1$, if$i=0$
- $d_H(C)=\rho_0+1)p^{\kappa_0}$, if $p^s-p^{s-\kappa_0}+(\rho_0-1)p^{s-\kappa_0-1}+1\le i\le p^s-p^{s-\kappa_0}+\rho_0 p^{s-\kappa_0-1}$
- $d_H(C)=0$, if $i=p^s$

where $1\le\rho_0\le p-1,\ 0\le\kappa_0\le s-1$.

In the following, we consider the Hamming distances of $(\alpha+u\beta)$-constacyclic codes for all structures in Table I.

## 3 Hamming Distances of $(\alpha+u\beta)$-constacyclic codes of length $4p^s$ over $R=\mathbb{F}_{p^m}+u\mathbb{F}_{p^m}$

**Structure I: $p^m\equiv 3(\mathrm{mod} 4)$ and $\lambda=(\alpha+u\beta)$ is a square in $R$.**

In this case, there exists $\alpha_0\in\mathbb{F}_{p^m}^*$ such that $\alpha=\alpha_0^{4p^s}$. Also, $(x^2+{\alpha_0}^2)$ is irreducible in $\mathbb{F}_{p^m}[x]$, and the factorization of $x^{4p^s}-\alpha$ into product of monic irreducible factors is $x^{4p^s}-\alpha=(x-\alpha_0)^{p^s}(x+\alpha_0)^{p^s}(x^2+{\alpha_0}^2)^{p^s}$.

Thus, the ring $R_{\alpha+u\beta}$ is a principal ideal ring whose ideals are $C=\langle (x-\alpha_0)^i(x+\alpha_0)^j(x^2+{\alpha_0}^2)^k\rangle$, where $0\le i,j,k\le 2p^s$. Equivalently, each $(\alpha+u\beta)$-constacyclic code of length $4p^s$ over $R$ has the form $C=\langle (x-\alpha_0)^i(x+\alpha_0)^j(x^2+{\alpha_0}^2)^k\rangle$, $0\le i,j,k\le 2p^s$. Then the following lemma follows:

**Lemma 3.1** In $R_{\alpha+u\beta}$, $\langle (x-\alpha_0)^{p^s}(x+\alpha_0)^{p^s}(x^2+{\alpha_0}^2)^{p^s}\rangle=\langle u\rangle$. In particular, $(x-\alpha_0)(x+\alpha_0)(x^2+{\alpha_0}^2)$ is nilpotent in $R_{\alpha+u\beta}$ with nilpotency index $2p^s$.

***Proof.*** In $R_{\alpha+u\beta}$, $(x-\alpha_0)^{p^s}(x+\alpha_0)^{p^s}(x^2+{\alpha_0}^2)^{p^s}=x^{4p^s}-{\alpha_0}^{4p^s}=\alpha+u\beta-\alpha=u\beta$. Thus, $\langle(x-\alpha_0)^{p^s}(x+\alpha_0)^{p^s}(x^2+{\alpha_0}^2)^{p^s}\rangle=\langle u\rangle$. Since $u$ has nilpotency index 2 in $R_{\alpha+u\beta}$, the last statement is straightforward. □

Now, we consider the Hamming distance of $C=\langle(x-\alpha_0)^i(x+\alpha_0)^j(x^2+{\alpha_0}^2)^k\rangle$ for the case $0\le k\le j\le i\le 2p^s$.

**Theorem 3.2** Let $1\le\rho_0,\rho_1,\rho_2\le p-1$, $0\le\kappa_2\le\kappa_1\le\kappa_0\le s-1$. Let $C=\langle(x-\alpha_0)^i(x+\alpha_0)^j(x^2+{\alpha_0}^2)^k\rangle\subseteq\frac{R[x]}{\langle x^{4p^s}-\lambda\rangle}$ for $0\le k\le j\le i\le 2p^s$ and $d_H(C)$ is determined by

- $d_H(C)=1$, if $0\le k\le j\le i\le p^s$
- $d_H(C)=2$, if $0\le k\le p^s$, $0\le j\le 2p^s$ and $p^s+1\le i\le 2p^s$
- $d_H(C)=\min\{(\rho_0+1)p^{\kappa_0},2(\rho_2+1)p^{\kappa_2}\}$, if $2p^s-p^{s-\kappa_0}+(\rho_0-1)p^{s-\kappa_0-1}+1\le i\le 2p^s-p^{s-\kappa_0}+\rho_0p^{s-\kappa_0-1}$, $2p^s-p^{s-\kappa_1}+(\rho_1-1)p^{s-\kappa_1-1}+1\le j\le 2p^s-p^{s-\kappa_1}+\rho_1p^{s-\kappa_1-1}$, $2p^s-p^{s-\kappa_2}+(\rho_2-1)p^{s-\kappa_2-1}+1\le k\le 2p^s-p^{s-\kappa_2}+\rho_2p^{s-\kappa_2-1}$
- $d_H(C)=2(\rho_2+1)p^{\kappa_2}$, if $i=2p^s$, $2p^s-p^{s-\kappa_1}+(\rho_1-1)p^{s-\kappa_1-1}+1\le j\le 2p^s$, $2p^s-p^{s-\kappa_2}+(\rho_2-1)p^{s-\kappa_2-1}+1\le k\le 2p^s-p^{s-\kappa_2}+\rho_2p^{s-\kappa_2-1}$
- $d_H(C)=0$, if $i=j=k=2p^s$

***Proof.***

**Case 1** $i=j=k=0$. Then, trivially $C$ has a Hamming distance of 1.

**Case 2** $j=k=0$ and $i\neq 0$.

**Subcase 2.1** $1\le i\le p^s$.

From 3.1, Clearly we have, $u\in\langle(x-\alpha_0)^i\rangle$. Thus $\langle(x-\alpha_0)^i\rangle$ has a Hamming distance of 1.

**Subcase 2.2** $p^s+1\le i\le 2p^s$.

Then, from 3.1 and subcase 2.1, we have $\langle(x-\alpha_0)^i\rangle\supseteq\langle u(x-\alpha_0)^{i-p^s}\rangle$.

So, $\langle(x-\alpha_0)^i\rangle$ has the same Hamming distances as the code $\langle(x-\alpha_0)^{i-p^s}\rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha\rangle}$, multiplied by $u$. Thus, the code has a Hamming distance of 2 from Theorem 2.4.

**Case 3** $k=0$ and $i\neq 0, j\neq 0$

**Subcase 3.1** $1\le j\le i\le p^s$.

Then, by 3.1, clearly, $u\in\langle(x-\alpha_0)^i(x+\alpha_0)^j\rangle$ and thus $\langle(x-\alpha_0)^i(x+\alpha_0)^j\rangle$ has a Hamming distance of 1.

**Subcase 3.2** $p^s+1\le j\le i\le 2p^s$.

Then $\langle(x-\alpha_0)^i(x+\alpha_0)^j\rangle\supseteq\langle u(x-\alpha_0)^{i-p^s}(x+\alpha_0)^{j-p^s}\rangle$

So, $\langle(x-\alpha_0)^i(x+\alpha_0)^j\rangle$ has the same Hamming distances as the code $\langle(x-\alpha_0)^{i-p^s}(x+\alpha_0)^{j-p^s}\rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha\rangle}$, multiplied by $u$. Thus, the code has a Hamming distance 2 from Theorem 2.4.

**Subcase 3.3** $p^s+1\le i\le 2p^s$ and $1\le j\le p^s$.

Then $\langle(x-\alpha_0)^i(x+\alpha_0)^j\rangle\supseteq\langle u(x-\alpha_0)^{i-p^s}\rangle$ by Subcase 3.1 and Lemma 3.1.

So, $\langle(x-\alpha_0)^i(x+\alpha_0)^j\rangle$ has the same Hamming distances as the code $\langle(x-\alpha_0)^{i-p^s}\rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha\rangle}$, multiplied by $u$. Thus, the code has a Hamming distance 2 from Theorem 2.4.

**Case 4** $i\neq 0, j\neq 0$ and $k\neq 0$

**Subcase 4.1** $1\le k\le j\le i\le p^s$.

Then, by 3.1, we have $u \in \langle (x-\alpha_0)^i (x+\alpha_0)^j (x^2+{\alpha_0}^2)^k \rangle$ and thus $\langle (x-\alpha_0)^i (x+\alpha_0)^j (x^2+{\alpha_0}^2)^k \rangle$ has a Hamming distance 1.

**Subcase 4.2** $p^s+1 \le k \le j \le i \le 2p^s-1$.

Then, $\langle (x-\alpha_0)^i (x+\alpha_0)^j (x^2+{\alpha_0}^2)^k \rangle = \langle u(x-\alpha_0)^{i-p^s} (x+\alpha_0)^{j-p^s} (x^2+{\alpha_0}^2)^{k-p^s} \rangle$.

So, the code $\langle (x-\alpha_0)^i (x+\alpha_0)^j (x^2+{\alpha_0}^2)^k \rangle$ in $R_{\alpha+u\beta}$ has the same Hamming distances as the code $\langle (x-\alpha_0)^{i-p^s} (x+\alpha_0)^{j-p^s} (x^2+{\alpha_0}^2)^{k-p^s} \rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha \rangle}$, multiplied by $u$. Thus, $C$ has the Hamming distances computed as

$d_H(C) = \min\{(\rho_0+1)p^{\kappa_0}, 2(\rho_2+1)p^{\kappa_2}\}$, if $2p^s - p^{s-\kappa_0} + (\rho_0-1)p^{s-\kappa_0-1} + 1 \le i \le 2p^s - p^{s-\kappa_0} + \rho_0 p^{s-\kappa_0-1}$, $2p^s - p^{s-\kappa_1} + (\rho_1-1)p^{s-\kappa_1-1} + 1 \le j \le 2p^s - p^{s-\kappa_1} + \rho_1 p^{s-\kappa_1-1}$, $2p^s - p^{s-\kappa_2} + (\rho_2-1)p^{s-\kappa_2-1} + 1 \le k \le 2p^s - p^{s-\kappa_2} + \rho_2 p^{s-\kappa_2-1}$ from Theorem 2.4.

**Subcase 4.3** $1 \le k \le p^s$ and $p^s+1 \le j \le i \le 2p^s$.

Then, clearly $\langle (x-\alpha_0)^i (x+\alpha_0)^j (x^2+{\alpha_0}^2)^k \rangle \supseteq \langle u(x-\alpha_0)^{i-p^s} (x+\alpha_0)^{j-p^s} \rangle$ by subcase 4.1. Thus, $C$ has a Hamming distance of 2 from Theorem 2.4.

**Subcase 4.4** $1 \le k \le j \le p^s$ and $p^s+1 \le i \le 2p^s$.

Then, clearly $\langle (x-\alpha_0)^i (x+\alpha_0)^j (x^2+{\alpha_0}^2)^k \rangle \supseteq \langle u(x-\alpha_0)^{i-p^s} \rangle$ by subcase 4.1. Thus, $C$ has a Hamming distance of 2 from Theorem 2.4.

**Subcase 4.5** $i = 2p^s$ and $p^s+1 \le k \le j \le 2p^s-1$.

Then, $\langle (x-\alpha_0)^i (x+\alpha_0)^j (x^2+{\alpha_0}^2)^k \rangle = \langle u(x-\alpha_0)^{p^s} (x+\alpha_0)^{j-p^s} (x^2+{\alpha_0}^2)^{k-p^s} \rangle$ by Lemma 2.1.

So, the code $\langle (x-\alpha_0)^i (x+\alpha_0)^j (x^2+{\alpha_0}^2)^k \rangle$ in $R_{\alpha+u\beta}$ has the same Hamming distance as the code $\langle (x-\alpha_0)^{p^s} (x+\alpha_0)^{j-p^s} (x^2+{\alpha_0}^2)^{k-p^s} \rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha \rangle}$, multiplied by $u$. Thus, $C$ has the Hamming distances of $2(\rho_2+1)p^{\kappa_2}$ from Theorem 2.4.

**Subcase 4.6** $i = j = 2p^s$ and $p^s+1 \le k \le 2p^s-1$.

Then, $\langle (x-\alpha_0)^i (x+\alpha_0)^j (x^2+{\alpha_0}^2)^k \rangle = \langle u(x-\alpha_0)^{p^s} (x+\alpha_0)^{p^s} (x^2+{\alpha_0}^2)^{k-p^s} \rangle$.

So, the code $\langle (x-\alpha_0)^i (x+\alpha_0)^j (x^2+{\alpha_0}^2)^k \rangle$ in $R_{\alpha+u\beta}$ has same Hamming distances as the code $\langle (x-\alpha_0)^{p^s} (x+\alpha_0)^{p^s} (x^2+{\alpha_0}^2)^{k-p^s} \rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha \rangle}$, multiplied by $u$. Thus , the Hamming distances computed as Theorem 2.4.

**Subcase 4.7** $i = j = k = 2p^s$, then $C$ has Hamming distance of 0.

Combining all the cases we get the Hamming distances of all $(\alpha+u\beta)$-constacyclic codes when $k \le j \le i$. ■

**Remark 3.3** Using the same technique as above, it is easy to check that the corresponding case with $k \le i \le j$ has the same Hamming distances as $k \le j \le i$ in the above Theorem.

Now, we consider the case $0 \le j \le k \le i \le 2p^s$. For this case, Hamming distances of $C$ is determined by the following Theorem.

**Theorem 3.4** Let $C = \langle (x-\alpha_0)^i (x+\alpha_0)^j (x^2+{\alpha_0}^2)^k \rangle \subseteq \frac{R[x]}{\langle x^{4p^s}-\alpha \rangle}$ for $0 \le j \le k \le i \le 2p^s$ and $d_H(C)$ is determined by

- $d_H(C) = 1$, if $0 \le j \le k \le i \le p^s$
- $d_H(C) = 2$, if $0 \le j \le k \le p^s$ and $p^s+1 \le i \le 2p^s$ or $0 \le j \le p^s, p^s+1 \le k \le p^s+p^{s-1}$ and $p^s+1 \le i \le p^s+p^{s-1}$
    - $d_H(C) = 3$, if $0 \le j \le p^s, p^s+1 \le k \le p^s+2p^{s-1}$ and $p^s+p^{s-1}+1 \le i \le p^s+2p^{s-1}$

- $d_H(C) = 4$, if $0 \le j \le p^s$, $p^s + 1 \le k \le 2p^s$ and $p^s + 2p^{s-1} + 1 \le i \le 2p^s$
- $d_H(C) = \min\{(\rho_0 + 1)p^{\kappa_0}, 2(\rho_1 + 1)p^{\kappa_1}, 4(\rho_2 + 1)p^{\kappa_2}\}$, if $2p^s - p^{s-\kappa_0} + (\rho_0 - 1)p^{s-\kappa_0-1} + 1 \le i \le 2p^s - p^{s-\kappa_0} + \rho_0 p^{s-\kappa_0-1}, 2p^s - p^{s-\kappa_1} + (\rho_1 - 1)p^{s-\kappa_1-1} + 1 \le k \le 2p^s - p^{s-\kappa_1} + \rho_1 p^{s-\kappa_1-1}, 2p^s - p^{s-\kappa_2} + (\rho_2 - 1)p^{s-\kappa_2-1} + 1 \le j \le 2p^s - p^{s-\kappa_2} + \rho_2 p^{s-\kappa_2-1}$
- $d_H(C) = \min\{(2(\rho_1 + 1)p^{\kappa_1}, 4(\rho_2 + 1)p^{\kappa_2}\}$, if $i = 2p^s, 2p^s - p^{s-\kappa_1} + (\rho_1 - 1)p^{s-\kappa_1-1} + 1 \le k \le 2p^s - p^{s-\kappa_1} + \rho_1 p^{s-\kappa_1-1}, 2p^s - p^{s-\kappa_2} + (\rho_2 - 1)p^{s-\kappa_2-1} + 1 \le j \le 2p^s - p^{s-\kappa_2} + \rho_2 p^{s-\kappa_2-1}$
- $d_H(C) = 4(\rho_2 + 1)p^{\kappa_2}$, If $i = k = 2p^s, 2p^s - p^{s-\kappa_2} + (\rho_2 - 1)p^{s-\kappa_2-1} + 1 \le j \le 2p^s - p^{s-\kappa_2} + \rho_2 p^{s-\kappa_2-1}$
- $d_H(C) = 0$, if $i = j = k = 2p^s$

where $1 \le \rho_0, \rho_1, \rho_2 \le p - 1$, $0 \le \kappa_2 \le \kappa_1 \le \kappa_0 \le s - 1$.

***Proof.***

**Case 1** $i = j = k = 0$. Then, $C$ has Hamming distance of 1.

**Case 2** $j = k = 0$ and $i \ne 0$.

**Subcase 2.1** $1 \le i \le p^s$.

Then, clearly $u \in \langle (x - \alpha_0)^i \rangle$. Thus $\langle (x - \alpha_0)^i \rangle$ has a Hamming distance of 1.

**Subcase 2.2** $p^s + 1 \le i \le 2p^s$.

Then, clearly $\langle (x - \alpha_0)^i \rangle \supseteq \langle u(x - \alpha_0)^{i-p^s} \rangle$ from subcase 2.1.

So, $\langle (x - \alpha_0)^i \rangle$ has the same Hamming distances as the code $\langle (x - \alpha_0)^{i-p^s} \rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s} - \alpha \rangle}$, multiplied by $u$. Thus, the code has Hamming distance of 2 from Theorem 2.6.

**Case 3** $j = 0$ and $i \ne 0, k \ne 0$

**Subcase 3.1** $1 \le k \le i \le p^s$.

Then, clearly $u \in \langle (x - \alpha_0)^i (x^2 + \alpha_0^2)^k \rangle$ and thus $\langle (x - \alpha_0)^i (x^2 + \alpha_0^2)^k \rangle$ has a Hamming distance of 1.

**Subcase 3.2** $p^s + 1 \le k \le i \le 2p^s$.

Then, we have $\langle (x - \alpha_0)^i (x^2 + \alpha_0^2)^k \rangle \supseteq \langle u(x - \alpha_0)^{i-p^s} (x^2 + \alpha_0^2)^{k-p^s} \rangle$.

So, the code $\langle (x - \alpha_0)^i (x^2 + \alpha_0^2)^k \rangle$ in $R_{\alpha+u\beta}$ has Hamming distances same as the code $\langle (x - \alpha_0)^{i-p^s} (x^2 + {\alpha^2}_0)^{k-p^s} \rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s} - \alpha \rangle}$, multiplied by $u$. Thus,

- $d_H(C) = 2$, if $j = 0$ , $p^s + 1 \le k \le i \le p^s + p^{s-1}$
- $d_H(C) = 3$, if $j = 0$, $p^s + 1 \le k \le p^s + 2p^{s-1}$ and $p^s + p^{s-1} + 1 \le i \le p^s + 2p^{s-1}$
- $d_H(C) = 4$, if $j = 0$ *and* $p^s + 1 \le k \le 2p^s$ and $p^s + 2p^{s-1} + 1 \le 2p^s$

from Theorem 2.6.

**Subcase 3.3** $1 \le k \le p^s$ and $p^s + 1 \le i \le 2p^s$.

Then, $\langle (x - \alpha_0)^i (x^2 + \alpha_0^2)^k \rangle \supseteq \langle u(x - \alpha_0)^{i-p^s} \rangle$.

So, the code $\langle (x - \alpha_0)^i (x^2 + \alpha_0^2)^k \rangle$ in $R_{\alpha+u\beta}$ has Hamming distance same as the code $\langle (x - \alpha_0)^{i-p^s} \rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s} - \alpha \rangle}$, multiplied by $u$. Thus, the code has a Hamming distance of 2 from Theorem 2.6.

**Case 4** $i \ne 0, j \ne 0$ and $k \ne 0$

**Subcase 4.1** $1 \le j \le k \le i \le p^s$.

Then, clearly $u \in \langle (x - \alpha_0)^i (x + \alpha_0)^j (x^2 + {\alpha_0}^2)^k \rangle$ and thus $\langle (x - \alpha_0)^i (x + \alpha_0)^j (x^2 + {\alpha_0}^2)^k \rangle$ has a

Hamming distance of 1.

**Subcase 4.2** $p^s + 1 \leq j \leq k \leq i \leq 2p^s - 1$.

Then, we have $\langle (x-\alpha_0)^i (x+\alpha_0)^j (x^2+{\alpha_0}^2)^k \rangle = \langle u(x-\alpha_0)^{i-p^s} (x+\alpha_0)^{j-p^s} (x^2+{\alpha_0}^2)^{k-p^s} \rangle$.

So, the code $\langle (x-\alpha_0)^i (x+\alpha_0)^j (x^2+{\alpha_0}^2)^k \rangle$ in $R_{\alpha+u\beta}$ has the same Hamming distances as the code $\langle (x-\alpha_0)^{i-p^s} (x+\alpha_0)^{j-p^s} \rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha \rangle}$, multiplied by $u$. Thus, Hamming distances computed as Theorem 2.6,

$$d_H(C) = \min\{(\rho_0+1)p^{\kappa_0}, 2(\rho_1+1)p^{\kappa_1}, 4(\rho_2+1)p^{\kappa_2}\}, \text{if } 2p^s - p^{s-\kappa_0} + (\rho_0-1)p^{s-\kappa_0-1} + 1 \leq i \leq 2p^s - p^{s-\kappa_0} + \rho_0 p^{s-\kappa_0-1}, 2p^s - p^{s-\kappa_1} + (\rho_1-1)p^{s-\kappa_1-1} + 1 \leq k \leq 2p^s - p^{s-\kappa_1} + \rho_1 p^{s-\kappa_1-1}, 2p^s - p^{s-\kappa_2} + (\rho_2-1)p^{s-\kappa_2-1} + 1 \leq j \leq 2p^s - p^{s-\kappa_2} + \rho_2 p^{s-\kappa_2-1}$$

**Subcase 4.3** $i = 2p^s$ and $p^s + 1 \leq j \leq k \leq 2p^s - 1$.

Then, we have $\langle (x-\alpha_0)^i (x+\alpha_0)^j (x^2+{\alpha_0}^2)^k \rangle = \langle u(x-\alpha_0)^{p^s} (x+\alpha_0)^{j-p^s} (x^2+{\alpha_0}^2)^{k-p^s} \rangle$.

So, the code $\langle (x-\alpha_0)^i (x+\alpha_0)^j (x^2+{\alpha_0}^2)^k \rangle$ in $R_{\alpha+u\beta}$ has the same Hamming distances as the code $\langle (x-\alpha_0)^{p^s} (x+\alpha_0)^{j-p^s} (x^2+{\alpha_0}^2)^{k-p^s} \rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha \rangle}$, multiplied by $u$. Thus, from Theorem 2.6, the Hamming distances computed as

$$d_H(C) = \min\{(2(\rho_1+1)p^{\kappa_1}, 4(\rho_2+1)p^{\kappa_2}\}, \text{if } i = 2p^s, 2p^s - p^{s-\kappa_1} + (\rho_1-1)p^{s-\kappa_1-1} + 1 \leq k \leq 2p^s - p^{s-\kappa_1} + \rho_1 p^{s-\kappa_1-1}, 2p^s - p^{s-\kappa_2} + (\rho_2-1)p^{s-\kappa_2-1} + 1 \leq j \leq 2p^s - p^{s-\kappa_2} + \rho_2 p^{s-\kappa_2-1}$$

**Subcase 4.4** $i = k = 2p^s$ and $p^s + 1 \leq j \leq 2p^s - 1$.

Then, $\langle (x-\alpha_0)^i (x+\alpha_0)^j (x^2+{\alpha_0}^2)^k \rangle = \langle u(x-\alpha_0)^{p^s} (x+\alpha_0)^{p^s} (x^2+{\alpha_0}^2)^{k-p^s} \rangle$.

So, the code $\langle (x-\alpha_0)^i (x+\alpha_0)^j (x^2+{\alpha_0}^2)^k \rangle$ in $R_{\alpha+u\beta}$ has same Hamming distances as the code $\langle (x-\alpha_0)^{p^s} (x+\alpha_0)^j (x^2+{\alpha_0}^2)^{p^s} \rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha \rangle}$, multiplied by $u$. Thus, the Hamming distances computed as

$d_H(C) = 4(\rho_2+1)p^{\kappa_2}, \text{if } i = k = 2p^s, 2p^s - p^{s-\kappa_2} + (\rho_2-1)p^{s-\kappa_2-1} + 1 \leq j \leq 2p^s - p^{s-\kappa_2} + \rho_2 p^{s-\kappa_2-1}$ from Theorem 2.6.

**Subcase 4.5** $1 \leq j \leq p^s$ and $p^s + 1 \leq k \leq i \leq 2p^s$.

Then, $\langle (x-\alpha_0)^i (x^2+\alpha_0^2)^k \rangle \supseteq \langle u(x-\alpha_0)^{i-p^s} (x^2+\alpha_0^2)^{k-p^s} \rangle$.

So, the code $\langle (x-\alpha_0)^i (x^2+\alpha_0^2)^k \rangle$ in $R_{\alpha+u\beta}$ has Hamming distance same as the code $\langle (x-\alpha_0)^{i-p^s} (x^2+\alpha_0^2)^{k-p^s} \rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha \rangle}$, multiplied by $u$. Thus, the Hamming distances computed as Theorem 2.6,

- $d_H(C) = 2, \text{if } 1 \leq j \leq p^s, p^s + 1 \leq k \leq i \leq p^s + p^{s-1}$
- $d_H(C) = 3, \text{if } 1 \leq j \leq p^s, \quad p^s + 1 \leq k \leq p^s + 2p^{s-1} \textit{ and } p^s + p^{s-1} + 1 \leq i \leq p^s + 2p^{s-1}$
- $d_H(C) = 4, \text{if } 1 \leq j \leq p^s \textit{ and } p^s + 1 \leq k \leq 2p^s \textit{ and } p^s + 2p^{s-1} + 1 \leq i \leq 2p^s$

**Subcase 4.6** $1 \leq j \leq k \leq p^s$ and $p^s + 1 \leq i \leq 2p^s$.

Then, $\langle (x-\alpha_0)^i (x^2+\alpha_0^2)^k \rangle \supseteq \langle u(x-\alpha_0)^{i-p^s} \rangle$.

So, the code $\langle (x-\alpha_0)^i (x^2+\alpha_0^2)^k \rangle$ in $R_{\alpha+u\beta}$ has Hamming distance same as the code $\langle (x-\alpha_0)^{i-p^s} \rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha \rangle}$, multiplied by $u$. Thus the code has a Hamming distance of 2 from Theorem 2.6.

**Subcase 4.7** $i = j = k = 2p^s$, then the code has Hamming distance of 0.

Combining all the cases we get the result.■

**Remark 3.5** Using the same technique as above, it is easy to check that the corresponding cases with $i \leq k \leq j, i \leq j \leq k, j \leq i \leq k$ have the same Hamming distances as case $j \leq k \leq i$ in the above Theorem.

**Structure II: $p^m \equiv 3(\mathrm{mod}4)$ and $\lambda = \alpha + u\beta$ is non-square unit in $R$.**

In this case, there exist $\alpha_0, \delta \in \mathbb{F}_{p^m}^*$ such that $\alpha = \alpha_0^{p^s}$ and $\alpha_0 = -4\delta^4$. And $x^{4p^s} - \alpha$ is factorized into product of irreducible factors as $x^{4p^s} - \alpha = (x^2 + 2\delta x + 2\delta^2)^{p^s}(x^2 - 2\delta x + 2\delta^2)^{p^s}$.

So, $C = \langle (x^2 + 2\delta x + 2\delta^2)^i (x^2 - 2\delta x + 2\delta^2)^j \rangle$, $0 \leq i, j \leq 2p^s$ are $(\alpha + u\beta)$-constacyclic codes of length $4p^s$ over $R$. Then, we have the following lemma:

**Lemma 3.6** In $R_{\alpha+u\beta}$, $\langle (x^2 + 2\delta x + 2\delta^2)^{p^s}(x^2 - 2\delta x + 2\delta^2)^{p^s} \rangle = \langle u \rangle$. In particular, $(x^2 + 2\delta x + 2\delta^2)(x^2 - 2\delta x + 2\delta^2)$ is nilpotent in $R_{\alpha+u\beta}$ with nilpotency index $2p^s$. Now, we consider the case $0 \leq j \leq i \leq 2p^s$. The following Theorem provide the Hamming distance of $C$ for this case.

**Theorem 3.7** The $(\alpha + u\beta)$-constacyclic codes of length $4p^s$ over $R$,

$$C = \langle (x^2 + 2\delta x + 2\delta^2)^i (x^2 - 2\delta x + 2\delta^2)^j \rangle$$

for $0 \leq j \leq i \leq 2p^s$ have the following Hamming distances

- $d_H(C) = 1$, if$0 \leq j \leq i \leq p^s$
- $d_H(C) = 2$, if $0 \leq j \leq p^s$ and $p^s + 1 \leq i \leq p^s + p^{s-1}$
- $d_H(C) = 3$, if $0 \leq j \leq p^s$ and $p^s + p^{s-1} + 1 \leq i \leq 2p^s$
- $d_H(C) = \min\{(\rho_0 + 1)p^{\kappa_0}, 3(\rho_1 + 1)p^{\kappa_1}\}$, if $2p^s - p^{s-\kappa_0} + (\rho_0 - 1)p^{s-\kappa_0-1} + 1 \leq i \leq 2p^s - p^{s-\kappa_0} + \rho_0 p^{s-\kappa_0-1}, 2p^s - p^{s-\kappa_1} + (\rho_1 - 1)p^{s-\kappa_1-1} + 1 \leq j \leq 2p^s - p^{s-\kappa_1} + \rho_1 p^{s-\kappa_1-1}$
- $d_H(C) = 3(\rho_1 + 1)p^{\kappa_1}$, if $i = 2p^s, 2p^s - p^{s-\kappa_1} + (\rho_1 - 1)p^{s-\kappa_1-1} + 1 \leq j \leq 2p^s - p^{s-\kappa_1} + \rho_1 p^{s-\kappa_1-1}$
- $d_H(C) = 0$, if $i = j = 2p^s$

where $1 \leq \rho_0, \rho_1 \leq p - 1$, $0 \leq \kappa_1 \leq \kappa_0 \leq s - 1$.

***Proof.***

**Case 1** $i = j = 0$. Then, the code $C$ has Hamming distance of 1.

**Case 2** $j = 0$ and $i \neq 0$.

**Subcase 2.1** $1 \leq i \leq p^s$.

Then, clearly from 3.6, we have $u \in \langle (x^2 + 2\delta x + 2\delta^2)^i \rangle$ and thus $\langle (x^2 + 2\delta x + 2\delta^2)^i \rangle$ has a Hamming distance of 1.

**Subcase 2.2** $p^s + 1 \leq i \leq 2p^s$.

Then, clearly $\langle (x^2 + 2\delta x + 2\delta^2)^i \rangle \supseteq \langle u(x^2 + 2\delta x + 2\delta^2)^{i-p^s} \rangle$.

So, $\langle (x^2 + 2\delta x + 2\delta^2)^i \rangle$ has the same Hamming distances as the code $\langle (x^2 + 2\delta x + 2\delta^2)^{i-p^s} \rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s} - \alpha \rangle}$, multiplied by $u$. Hence, from Theorem 2.8, we have

- $d_H(C) = 2$, if $j = 0$ and $p^s + 1 \leq i \leq p^s + p^{s-1}$
- $d_H(C) = 3$, if $j = 0$ and $p^s + p^{s-1} + 1 \leq i \leq 2p^s$

**Case 3** $i \neq 0, j \neq 0$

**Subcase 3.1** $1 \leq j \leq i \leq p^s$.

Then, $u \in \langle (x^2 + 2\delta x + 2\delta^2)^i (x^2 - 2\delta x + 2\delta^2)^j \rangle$ and thus $\langle (x^2 + 2\delta x + 2\delta^2)^i (x^2 - 2\delta x + 2\delta^2)^j \rangle$ has a Hamming distance of 1.

**Subcase 3.2** $p^s + 1 \leq j \leq i \leq 2p^s - 1$.

Then $\langle (x^2+2\delta x+2\delta^2)^i(x^2-2\delta x+2\delta^2)^j\rangle = \langle u(x^2+2\delta x+2\delta^2)^{i-p^s}(x^2-2\delta x+2\delta^2)^{j-p^s}\rangle$.
So, the code $\langle (x^2+2\delta x+2\delta^2)^i(x^2-2\delta x+2\delta^2)^j\rangle$ in $R_{\alpha+u\beta}$ has the Hamming distances same as the code $\langle (x^2+2\delta x+2\delta^2)^{i-p^s}(x^2-2\delta x+2\delta^2)^{j-p^s}\rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha\rangle}$, multiplied by $u$. Thus, from Theorem 2.8, the Hamming distances computed as

$$d_H(C) = \min\{(\rho_0+1)p^{\kappa_0}, 3(\rho_1+1)p^{\kappa_1}\}, \text{if } 2p^s-p^{s-\kappa_0}+(\rho_0-1)p^{s-\kappa_0-1}+1 \le i \le 2p^s-p^{s-\kappa_0}+\rho_0 p^{s-\kappa_0-1}, 2p^s-p^{s-\kappa_1}+(\rho_1-1)p^{s-\kappa_1-1}+1 \le j \le 2p^s-p^{s-\kappa_1}+\rho_1 p^{s-\kappa_1-1}$$

**Subcase 3.3** $i = 2p^s, p^s+1 \le j \le 2p^s-1$.
Then, $\langle (x^2-2\delta x+2\delta^2)^i(x^2+2\delta x+2\delta^2)^j\rangle = \langle u(x^2+2\delta x+2\delta^2)^{p^s}(x^2-2\delta x+2\delta^2)^{j-p^s}\rangle$.
So, the code $\langle (x^2+2\delta x+2\delta^2)^i(x^2-2\delta x+2\delta^2)^j\rangle$ in $R_{\alpha+u\beta}$ has the same Hamming distance as the code $\langle (x^2+2\delta x+2\delta^2)^{p^s}(x^2-2\delta x+2\delta^2)^{j-p^s}\rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha\rangle}$, multiplied by $u$. Thus, from Theorem 2.8, the Hamming distances computed as

$$d_H(C) = 3(\rho_1+1)p^{\kappa_1}, \text{if } i = 2p^s, 2p^s-p^{s-\kappa_1}+(\rho_1-1)p^{s-\kappa_1-1}+1 \le j \le 2p^s-p^{s-\kappa_1}+\rho_1 p^{s-\kappa_1-1}$$

**Subcase 3.4** $1 \le j \le p^s$ and $p^s+1 \le i \le 2p^s$.
Then, $\langle (x^2+2\delta x+2\delta^2)^i(x^2-2\delta x+2\delta^2)^j\rangle \supseteq \langle u(x^2+2\delta x+2\delta^2)^{i-p^s}\rangle$.
So, $\langle (x^2+2\delta x+2\delta^2)^i(x^2-2\delta x+2\delta^2)^j\rangle$ has the same Hamming distances as the code $\langle (x^2+2\delta x+2\delta^2)^{i-p^s}\rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha\rangle}$, multiplied by $u$. Thus $\langle (x^2+2\delta x+2\delta^2)^i(x^2-2\delta x+2\delta^2)^j\rangle$ has a Hamming distance

- $d_H(C) = 2$, if $1 \le j \le p^s$ and $p^s+1 \le i \le p^s+p^{s-1}$
- $d_H(C) = 3$, if $1 \le j \le p^s$ and $p^s+p^{s-1}+1 \le i \le 2p^s$

**Subcase 3.5** $i = j = 2p^s$
Then, $\langle (x^2+2\delta x+2\delta^2)^i(x^2-2\delta x+2\delta^2)^j\rangle$ has Hamming distance 0.
Combining all the cases we get the Hamming distances of all $(\alpha+u\beta)$-constacyclic codes when $j \le i$.
■

**Remark 3.8** Using the same technique as above, it is easy to check that the corresponding case with $i \le j$ has the same Hamming distances as $j \le i$ in the above Theorem.

**Structure III: $p^m \equiv 1 (\mathrm{mod} 4)$ and $\lambda = \alpha + u\beta$ is a square of the form $\lambda = \lambda_0^4$ in $R$.**

Then, there exist $\alpha_0, \eta \in \mathbb{F}_{p^m}^*$ such that $\alpha = \alpha_0^{4p^s}$ and $\eta^2 = -1$. $x^{4p^s}-\alpha$ is factorized into product of irreducible factors as $x^{4p^s}-\alpha = (x-\alpha_0)^{p^s}(x+\alpha_0)^{p^s}(x-\eta\alpha_0)^{p^s}(x+\eta\alpha_0)^{p^s}$.
Let $C = \langle (x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle$, $0 \le i,j,k,l \le 2p^s$ be a $(\alpha+u\beta)$-constacyclic code of length $4p^s$ over $R$. Then, the following lemma follows:

**Lemma 3.9** In $R_{\alpha+u\beta}$, $\langle (x-\alpha_0)^{p^s}(x+\alpha_0)^{p^s}(x-\eta\alpha_0)^{p^s}(x+\eta\alpha_0)^{p^s}\rangle = \langle u\rangle$. In particular, $(x-\alpha_0)(x+\alpha_0)(x-\eta\alpha_0)(x+\eta\alpha_0)$ is nilpotent in $R_{\alpha+u\beta}$ with nilpotency index $2p^s$. Here, we consider the case $0 \le l \le k \le j \le i \le 2p^s$ to compute the Hamming distance of $C$.

**Theorem 3.10** Let $1 \le \rho_0, \rho_1, \rho_2, \rho_3 \le p-1$, $0 \le \kappa_3 \le \kappa_2 \le \kappa_1 \le \kappa_0 \le s-1$. Let $C = \langle (x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle \subseteq \frac{R[x]}{\langle x^{4p^s}-\lambda\rangle}$ for $0 \le l \le k \le j \le i \le 2p^s$ and $d_H(C)$ is determined by

- $d_H(C) = 1,$ if $0 \leq l \leq k \leq j \leq i \leq p^s$
- $d_H(C) = 2,$ if $0 \leq l \leq k \leq p^s, 0 \leq j \leq 2p^s$ and $p^s + 1 \leq i \leq 2p^s$ or $0 \leq l \leq p^s$ and $p^s + 1 \leq k \leq j \leq i \leq p^s + p^{s-1}$
- $d_H(C) = 4,$ if $0 \leq l \leq p^s, \quad p^s + 1 \leq k \leq j \leq 2p^s$ and $p^s + p^{s-1} + 1 \leq i \leq 2p^s$
- $d_H(C) = \min\{(\rho_0 + 1)p^{\kappa_0}, 2(\rho_2 + 1)p^{\kappa_2}, 4(\rho_3 + 1)p^{\kappa_3}\},$ if $2p^s - p^{s-\kappa_0} + (\rho_0 - 1)p^{s-\kappa_0-1} + 1 \leq i \leq 2p^s - p^{s-\kappa_0} + \rho_0 p^{s-\kappa_0-1}, 2p^s - p^{s-\kappa_1} + (\rho_1 - 1)p^{s-\kappa_1-1} + 1 \leq j \leq 2p^s - p^{s-\kappa_1} + \rho_1 p^{s-\kappa_1-1}, 2p^s - p^{s-\kappa_2} + (\rho_2 - 1)p^{s-\kappa_2-1} + 1 \leq k \leq 2p^s - p^{s-\kappa_2} + \rho_2 p^{s-\kappa_2-1}, 2p^s - p^{s-\kappa_3} + (\rho_3 - 1)p^{s-\kappa_3-1} + 1 \leq l \leq 2p^s - p^{s-\kappa_3} + \rho_3 p^{s-\kappa_3-1}$
- $d_H(C) = \min\{2(\rho_2 + 1)p^{\kappa_2}, 4(\rho_3 + 1)p^{\kappa_3}\},$ if $i = 2p^s, 2p^s - p^{s-\kappa_1} + (\rho_1 - 1)p^{s-\kappa_1-1} + 1 \leq j \leq 2p^s, 2p^s - p^{s-\kappa_2} + (\rho_2 - 1)p^{s-\kappa_2-1} + 1 \leq k \leq 2p^s - p^{s-\kappa_2} + \rho_2 p^{s-\kappa_2-1}, 2p^s - p^{s-\kappa_3} + (\rho_3 - 1)p^{s-\kappa_3-1} + 1 \leq l \leq 2p^s - p^{s-\kappa_3} + \rho_3 p^{s-\kappa_3-1}$
- $d_H(C) = 4(\rho_3 + 1)p^{\kappa_3},$ if $i = j = k = 2p^s, 2p^s - p^{s-\kappa_3} + (\rho_3 - 1)p^{s-\kappa_3-1} + 1 \leq l \leq 2p^s - p^{s-\kappa_3} + \rho_3 p^{s-\kappa_3-1}$
- $d_H(C) = 0,$ if $i = j = k = l = 2p^s$

***Proof.***

**Case 1** $i = j = k = l = 0$, then $C$ has a Hamming distance of 1.

**Case 2** $j = k = l = 0$ and $i \neq 0$.

**Subcase 2.1** $1 \leq i \leq p^s$.

From Lemma 3.9, Clearly we have, $u \in \langle (x - \alpha_0)^i \rangle$.Thus $\langle (x - \alpha_0)^i \rangle$ has a Hamming distance of 1.

**Subcase 2.2** $p^s + 1 \leq i \leq 2p^s$.

Then, from Lemma 3.9 and subcase 2.1 clearly, we have $\langle (x - \alpha_0)^i \rangle \supseteq \langle u(x - \alpha_0)^{i-p^s} \rangle$.

So, $\langle (x - \alpha_0)^i \rangle$ has the same Hamming distances as the code $\langle (x - \alpha_0)^{i-p^s} \rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s} - \alpha \rangle}$, multiplied by $u$. Thus, the code has a Hamming distance of 2 from Theorem 2.10.

**Case 3** $k = l = 0$ and $i \neq 0, j \neq 0$

**Subcase 3.1** $1 \leq j \leq i \leq p^s$.

Then, by Lemma 3.9, $u \in \langle (x - \alpha_0)^i (x + \alpha_0)^j \rangle$ and thus $\langle (x - \alpha_0)^i (x + \alpha_0)^j \rangle$ has a Hamming distance of 1.

**Subcase 3.2** $p^s + 1 \leq j \leq i \leq 2p^s$.

Then, $\langle (x - \alpha_0)^i (x + \alpha_0)^j \rangle \supseteq \langle u(x - \alpha_0)^{i-p^s} (x + \alpha_0)^{j-p^s} \rangle$

So, $\langle (x - \alpha_0)^i (x + \alpha_0)^j \rangle$ has the same Hamming distances as the code $\langle (x - \alpha_0)^{i-p^s} (x + \alpha_0)^{j-p^s} \rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s} - \alpha \rangle}$, multiplied by $u$. Thus, the code has a Hamming distance 2 from Theorem 2.10.

**Subcase 3.3** $p^s + 1 \leq i \leq 2p^s$ and $1 \leq j \leq p^s$.

Then $\langle (x - \alpha_0)^i (x + \alpha_0)^j \rangle \supseteq \langle u(x - \alpha_0)^{i-p^s} \rangle$ by Subcase 3.1 and Lemma 3.9.

So, $\langle (x - \alpha_0)^i (x + \alpha_0)^j \rangle$ has the same Hamming distances as the code $\langle (x - \alpha_0)^{i-p^s} \rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s} - \alpha \rangle}$, multiplied by $u$. Thus, the code has a Hamming distance 2 from Theorem 2.10.

**Case 4** $l = 0, i \neq 0, j \neq 0$ and $k \neq 0$

**Subcase 4.1** $1 \leq k \leq j \leq i \leq p^s$.

Then, by Lemma 3.9, we have $u \in \langle (x - \alpha_0)^i (x + \alpha_0)^j (x - \eta\alpha_0)^k \rangle$ and thus $\langle (x - \alpha_0)^i (x + \alpha_0)^j (x -$

$\eta\alpha_0)^k\rangle$ has a Hamming distance 1.

**Subcase 4.2** $p^s+1 \le k \le j \le i \le 2p^s$.

Then, $\langle (x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k\rangle \supseteq \langle u(x-\alpha_0)^{i-p^s}(x+\alpha_0)^{j-p^s}(x-\eta\alpha_0)^{k-p^s}\rangle$.

So, the code $\langle (x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k\rangle$ in $R_{\alpha+u\beta}$ has the same Hamming distances as the code $\langle (x-\alpha_0)^{i-p^s}(x+\alpha_0)^{j-p^s}(x-\eta\alpha_0)^{k-p^s}\rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha\rangle}$, multiplied by $u$. Thus, $C$ has the Hamming distances computed as

- $d_H(C)=2$, if $l=0$, $p^s+1 \le k \le j \le i \le p^s+p^{s-1}$
- $d_H(C)=4$, if $l=0$, $p^s+1 \le k \le j \le 2p^s$ and $p^s+p^{s-1}+1 \le i \le 2p^s$

from Theorem 2.10.

**Subcase 4.3** $1 \le k \le p^s$ and $p^s+1 \le j \le i \le 2p^s$.

Then, clearly $\langle (x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k\rangle \supseteq \langle u(x-\alpha_0)^{i-p^s}(x+\alpha_0)^{j-p^s}\rangle$ by subcase 4.1. Thus, $C$ has a Hamming distance of 2 from Theorem 2.10.

**Subcase 4.4** $1 \le k \le j \le p^s$ and $p^s+1 \le i \le 2p^s$.

Then, clearly $\langle (x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k\rangle \supseteq \langle u(x-\alpha_0)^{i-p^s}\rangle$ by subcase 4.1. Thus, $C$ has a Hamming distance of 2 from Theorem 2.10.

**Case 5** $i \ne 0, j \ne 0, k \ne 0$ and $l \ne 0$.

**Subcase 5.1** $1 \le l \le k \le j \le i \le p^s$.

Then, we have $u \in \langle (x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle$ and thus $\langle (x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle$ has a Hamming distance 1.

**Subcase 5.2** $p^s+1 \le l \le k \le j \le i \le 2p^s-1$.

We have, $\langle (x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle = \langle u(x-\alpha_0)^{i-p^s}(x+\alpha_0)^{j-p^s}(x-\eta\alpha_0)^{k-p^s}(x+\eta\alpha_0)^{l-p^s}\rangle$. So, the code $\langle (x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle$ in $R_{\alpha+u\beta}$ has the same Hamming distances as the code $\langle (x-\alpha_0)^{i-p^s}(x+\alpha_0)^{j-p^s}(x-\eta\alpha_0)^{k-p^s}(x+\eta\alpha_0)^{l-p^s}\rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha\rangle}$, multiplied by $u$. Thus, $C$ has the Hamming distances computed as

$$\begin{aligned} d_H(C) = \min\{(\rho_0+1)p^{\kappa_0}, 2(\rho_2+1)p^{\kappa_2}, 4(\rho_3+1)p^{\kappa_3}\}, &\text{if } 2p^s-p^{s-\kappa_0}+(\rho_0-1)p^{s-\kappa_0-1}+1 \le i \\ &\le 2p^s-p^{s-\kappa_0}+\rho_0 p^{s-\kappa_0-1}, 2p^s-p^{s-\kappa_1}+(\rho_1-1)p^{s-\kappa_1-1}+1 \le j \\ &\le 2p^s-p^{s-\kappa_1}+\rho_1 p^{s-\kappa_1-1}, 2p^s-p^{s-\kappa_2}+(\rho_2-1)p^{s-\kappa_2-1}+1 \le k \\ &\le 2p^s-p^{s-\kappa_2}+\rho_2 p^{s-\kappa_2-1}, 2p^s-p^{s-\kappa_3}+(\rho_3-1)p^{s-\kappa_3-1}+1 \le l \\ &\le 2p^s-p^{s-\kappa_3}+\rho_3 p^{s-\kappa_3-1} \end{aligned}$$

**Subcase 5.3** $i=2p^s$ and $p^s+1 \le l \le k \le j \le 2p^s-1$.

We have, $\langle (x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle = \langle u(x-\alpha_0)^{p^s}(x+\alpha_0)^{j-p^s}(x-\eta\alpha_0)^{k-p^s}(x+\eta\alpha_0)^{l-p^s}\rangle$. So, the code $\langle (x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle$ in $R_{\alpha+u\beta}$ has the same Hamming distances as the code $\langle (x-\alpha_0)^{p^s}(x+\alpha_0)^{j-p^s}(x-\eta\alpha_0)^{k-p^s}(x+\eta\alpha_0)^{l-p^s}\rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha\rangle}$, multiplied by $u$. Thus, $C$ has the Hamming distances computed as

$$\begin{aligned} d_H(C) = \min\{2(\rho_2+1)p^{\kappa_2}, 4(\rho_3+1)p^{\kappa_3}\}, &\text{If } i=2p^s, 2p^s-p^{s-\kappa_1}+(\rho_1-1)p^{s-\kappa_1-1}+1 \le j \\ &\le 2p^s-p^{s-\kappa_1}+\rho_1 p^{s-\kappa_1-1}, 2p^s-p^{s-\kappa_2}+(\rho_2-1)p^{s-\kappa_2-1}+1 \le k \\ &\le 2p^s-p^{s-\kappa_2}+\rho_2 p^{s-\kappa_2-1}, 2p^s-p^{s-\kappa_3}+(\rho_3-1)p^{s-\kappa_3-1}+1 \le l \\ &\le 2p^s-p^{s-\kappa_3}+\rho_3 p^{s-\kappa_3-1} \end{aligned}$$

**Subcase 5.4** $i=j=2p^s$ and $p^s+1 \le l \le k \le 2p^s-1$.

We have, $\langle(x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle = \langle u(x-\alpha_0)^{p^s}(x+\alpha_0)^{p^s}(x-\eta\alpha_0)^{k-p^s}(x+\eta\alpha_0)^{l-p^s}\rangle$. So, the code $\langle(x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle$ in $R_{\alpha+u\beta}$ has the same Hamming distances as the code $\langle(x-\alpha_0)^{p^s}(x+\alpha_0)^{p^s}(x-\eta\alpha_0)^{k-p^s}(x+\eta\alpha_0)^{l-p^s}\rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha\rangle}$, multiplied by $u$. Thus, $C$ has the Hamming distances computed as

$$d_H(C) = \min\{2(\rho_2+1)p^{\kappa_2}, 4(\rho_3+1)p^{\kappa_3}\}, \text{if } i=j=2p^s, 2p^s-p^{s-\kappa_2}+(\rho_2-1)p^{s-\kappa_2-1}+1 \le k \le 2p^s-p^{s-\kappa_2}+\rho_2 p^{s-\kappa_2-1}, 2p^s-p^{s-\kappa_3}+(\rho_3-1)p^{s-\kappa_3-1}+1 \le l \le 2p^s-p^{s-\kappa_3}+\rho_3 p^{s-\kappa_3-1}$$

**Subcase 5.5** $i=j=k=2p^s$ and $p^s+1 \le l \le 2p^s-1$.

We have, $\langle(x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle = \langle u(x-\alpha_0)^{p^s}(x+\alpha_0)^{p^s}(x-\eta\alpha_0)^{p^s}(x+\eta\alpha_0)^{l-p^s}\rangle$. So, the code $\langle(x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle$ in $R_{\alpha+u\beta}$ has the same Hamming distances as the code $\langle(x-\alpha_0)^{p^s}(x+\alpha_0)^{p^s}(x-\eta\alpha_0)^{p^s}(x+\eta\alpha_0)^{l-p^s}\rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha\rangle}$, multiplied by $u$. Thus, $C$ has the Hamming distances computed as

$$d_H(C) = 4(\rho_3+1)p^{\kappa_3}, \text{if } i=j=k=2p^s, 2p^s-p^{s-\kappa_3}+(\rho_3-1)p^{s-\kappa_3-1}+1 \le l \le 2p^s-p^{s-\kappa_3}+\rho_3 p^{s-\kappa_3-1}$$

**Subcase 5.6** $1 \le l \le p^s$ and $p^s+1 \le k \le j \le i \le 2p^s$.

We have, $\langle(x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle \supseteq \langle u(x-\alpha_0)^{i-p^s}(x+\alpha_0)^{j-p^s}(x-\eta\alpha_0)^{k-p^s}\rangle$. So, the code $\langle(x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle$ in $R_{\alpha+u\beta}$ has the same Hamming distances as the code $\langle(x-\alpha_0)^{i-p^s}(x+\alpha_0)^{j-p^s}(x-\eta\alpha_0)^{k-p^s}\rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha\rangle}$, multiplied by $u$. Thus, $C$ has the Hamming distances computed as

- $d_H(C) = 2, \text{if } l=0, \quad p^s+1 \le k \le j \le i \le p^s+p^{s-1}$
- $d_H(C) = 4, \text{if } l=0, \quad p^s+1 \le k \le j \le 2p^s \text{ and } p^s+p^{s-1}+1 \le i \le 2p^s$

from Theorem 2.10.

**Subcase 5.7** $1 \le l \le k \le p^s$ and $p^s+1 \le j \le i \le 2p^s$.

We have, $\langle(x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle \supseteq \langle u(x-\alpha_0)^{i-p^s}(x+\alpha_0)^{j-p^s}\rangle$. So, the code $\langle(x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle$ in $R_{\alpha+u\beta}$ has the same Hamming distances as the code $\langle(x-\alpha_0)^{i-p^s}(x+\alpha_0)^{j-p^s}\rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha\rangle}$, multiplied by $u$. Thus, $C$ has the Hamming distances computed as 2 from Theorem 2.10.

**Subcase 5.8** $1 \le l \le k \le j \le p^s$ and $p^s+1 \le i \le 2p^s$.

We have, $\langle(x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle \supseteq \langle u(x-\alpha_0)^{i-p^s}\rangle$. So, the code $\langle(x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle$ in $R_{\alpha+u\beta}$ has the same Hamming distances as the code $\langle(x-\alpha_0)^{i-p^s}\rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha\rangle}$, multiplied by $u$. Thus, $C$ has the Hamming distances computed as 2 from Theorem 2.10.

**Subcase 5.9** $i=j=k=l=2p^s$.

Then, $C$ has Hamming distance of 0.

Combining all the cases we get the Hamming distances of all $(\alpha+u\beta)$-constacyclic codes when $l \le k \le j \le i$. ■

**Remark 3.11** Using the same technique as above, it is easy to check that the corresponding cases with $k \le l \le j \le i$, $k \le l \le i \le j$, $l \le k \le i \le j$, $j \le i \le l \le k$, $i \le j \le k \le l$, $i \le j \le l \le k$ and $j \le i \le k \le l$ have the same Hamming distances as $l \le k \le j \le i$ in the above Theorem.

Next, we consider the Hamming distance of $C$ for the case $0 \le l \le j \le k \le i \le 2p^s$.

**Theorem 3.12** Let $C = \langle (x-\alpha_0)^i (x+\alpha_0)^j (x-\eta\alpha_0)^k (x+\eta\alpha_0)^l \rangle \subseteq \frac{R[x]}{\langle x^{4p^s}-\alpha \rangle}$ for $0 \le l \le j \le k \le i \le 2p^s$ and $d_H(C)$ is determined by

- $d_H(C) = 1$, if $0 \le l \le j \le k \le i \le p^s$
- $d_H(C) = 2$, if $0 \le l \le j \le k \le p^s$ and $p^s+1 \le i \le 2p^s$ or $0 \le l \le l$ and $p^s+1 \le j \le k \le i \le p^s+p^{s-1}$
- $d_H(C) = 3$, if $0 \le l \le j \le p^s$, $p^s+1 \le k \le 2p^s$ and $p^s+p^{s-1}+1 \le i \le 2p^s$ or $0 \le l \le p^s$, $p^s+1 \le j \le k \le p^s+2p^{s-1}$ and $p^s+p^{s-1}+1 \le i \le p^s+2p^{s-1}$
- $d_H(C) = 4$, if $0 \le l \le p^s$, $p^s+1 \le j \le k \le 2p^s$ *and* $p^s+2p^{s-1}+1 \le i \le 2p^s$
- $d_H(C) = \min\{(\rho_0+1)p^{\kappa_0}, 2(\rho_1+1)p^{\kappa_1}, 3(\rho_2+1)p^{\kappa_2}, 4(\rho_3+1)p^{\kappa_3}\}$, if $2p^s - p^{s-\kappa_0} + (\rho_0-1)p^{s-\kappa_0-1}+1 \le i \le 2p^s - p^{s-\kappa_0} + \rho_0 p^{s-\kappa_0-1}$, $2p^s - p^{s-\kappa_1} + (\rho_1-1)p^{s-\kappa_1-1}+1 \le j \le 2p^s - p^{s-\kappa_1} + \rho_1 p^{s-\kappa_1-1}$, $2p^s - p^{s-\kappa_2} + (\rho_2-1)p^{s-\kappa_2-1}+1 \le k \le 2p^s - p^{s-\kappa_2} + \rho_2 p^{s-\kappa_2-1}$, $2p^s - p^{s-\kappa_3} + (\rho_3-1)p^{s-\kappa_3-1}+1 \le l \le 2p^s - p^{s-\kappa_3} + \rho_3 p^{s-\kappa_3-1}$
- $d_H(C) = \min\{2(\rho_1+1)p^{\kappa_1}, 3(\rho_2+1)p^{\kappa_2}, 4(\rho_3+1)p^{\kappa_3}\}$, if $i = 2p^s$, $2p^s - p^{s-\kappa_1} + (\rho_1-1)p^{s-\kappa_1-1}+1 \le j \le 2p^s - p^{s-\kappa_1} + \rho_1 p^{s-\kappa_1-1}$, $2p^s - p^{s-\kappa_2} + (\rho_2-1)p^{s-\kappa_2-1}+1 \le k \le 2p^s - p^{s-\kappa_2} + \rho_2 p^{s-\kappa_2-1}$, $2p^s - p^{s-\kappa_3} + (\rho_3-1)p^{s-\kappa_3-1}+1 \le l \le 2p^s - p^{s-\kappa_3} + \rho_3 p^{s-\kappa_3-1}$
- $d_H(C) = \min\{3(\rho_2+1)p^{\kappa_2}, 4(\rho_3+1)p^{\kappa_3}\}$, if $i = j = 2p^s$, $2p^s - p^{s-\kappa_2} + (\rho_2-1)p^{s-\kappa_2-1}+1 \le k \le 2p^s - p^{s-\kappa_2} + \rho_2 p^{s-\kappa_2-1}$, $2p^s - p^{s-\kappa_3} + (\rho_3-1)p^{s-\kappa_3-1}+1 \le l \le 2p^s - p^{s-\kappa_3} + \rho_3 p^{s-\kappa_3-1}$
- $d_H(C) = 4(\rho_3+1)p^{\kappa_3}$, if $i = j = k = 2p^s$, $2p^s - p^{s-\kappa_3} + (\rho_3-1)p^{s-\kappa_3-1}+1 \le l \le 2p^s - p^{s-\kappa_3} + \rho_3 p^{s-\kappa_3-1}$
- $d_H(C) = 0$, if $i = j = k = l = 2p^s$

where $1 \le \rho_0, \rho_1, \rho_2, \rho_3 \le p-1$, $0 \le \kappa_3 \le \kappa_2 \le \kappa_1 \le \kappa_0 \le s-1$.

***Proof.***

**Case 1** $i = j = k = l = 0$. Then, $C$ has a Hamming distance of 1.

**Case 2** $j = k = l = 0$ and $i \ne 0$.

**Subcase 2.1** $1 \le i \le p^s$.

From Lemma 3.9, Clearly we have, $u \in \langle (x-\alpha_0)^i \rangle$.Thus $\langle (x-\alpha_0)^i \rangle$ has a Hamming distance of 1.

**Subcase 2.2** $p^s+1 \le i \le 2p^s$.

Then, from clearly, we have $\langle (x-\alpha_0)^i \rangle \supseteq \langle u(x-\alpha_0)^{i-p^s} \rangle$.

So, $\langle (x-\alpha_0)^i \rangle$ has the same Hamming distances as the code $\langle (x-\alpha_0)^{i-p^s} \rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha \rangle}$, multiplied by $u$. Thus, the code has a Hamming distance of 2 from Theorem 2.12.

**Case 3** $l = j = 0$ and $i \ne 0, k \ne 0$

**Subcase 3.1** $1 \le k \le i \le p^s$.

Then, by Lemma 3.9, clearly, $u \in \langle (x-\alpha_0)^i (x-\eta\alpha_0)^k \rangle$ and thus $\langle (x-\alpha_0)^i (x-\eta\alpha_0)^k \rangle$ has a Hamming distance of 1.

**Subcase 3.2** $p^s+1 \le k \le i \le 2p^s$.

Then $\langle (x-\alpha_0)^i (x-\eta\alpha_0)^k \rangle \supseteq \langle u(x-\alpha_0)^{i-p^s} (x-\eta\alpha_0)^{k-p^s} \rangle$

So, $\langle (x-\alpha_0)^i (x-\eta\alpha_0)^k \rangle$ has the same Hamming distances as the code $\langle (x-\alpha_0)^{i-p^s}(x-\eta\alpha_0)^{k-p^s} \rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha \rangle}$, multiplied by $u$. Thus, the code has a Hamming distance 3 from Theorem 2.12.

**Subcase 3.3** $p^s+1 \le i \le 2p^s$ and $1 \le k \le p^s$.

Then $\langle (x-\alpha_0)^i (x-\eta\alpha_0)^k \rangle \supseteq \langle u(x-\alpha_0)^{i-p^s} \rangle$.

So, $\langle (x-\alpha_0)^i (x-\eta\alpha_0)^k \rangle$ has the same Hamming distances as the code $\langle (x-\alpha_0)^{i-p^s} \rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha \rangle}$, multiplied by $u$. Thus, the code has a Hamming distance 2 from Theorem 2.12. **Case 4** $l=0, i \ne 0, j \ne 0$ and $k \ne 0$

**Subcase 4.1** $1 \le k \le j \le i \le p^s$.

Then, by Lemma 3.9, we have $u \in \langle (x-\alpha_0)^i (x+\alpha_0)^j (x-\eta\alpha_0)^k \rangle$ and thus $\langle (x-\alpha_0)^i (x+\alpha_0)^j (x-\eta\alpha_0)^k \rangle$ has a Hamming distance 1.

**Subcase 4.2** $p^s+1 \le k \le j \le i \le 2p^s$.

Then, $\langle (x-\alpha_0)^i (x+\alpha_0)^j (x-\eta\alpha_0)^k \rangle \supseteq \langle u(x-\alpha_0)^{i-p^s}(x+\alpha_0)^{j-p^s}(x-\eta\alpha_0)^{k-p^s} \rangle$.

So, the code $\langle (x-\alpha_0)^i (x+\alpha_0)^j (x-\eta\alpha_0)^k \rangle$ in $R_{\alpha+u\beta}$ has the same Hamming distances as the code $\langle (x-\alpha_0)^{i-p^s}(x+\alpha_0)^{j-p^s}(x-\eta\alpha_0)^{k-p^s} \rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha \rangle}$, multiplied by $u$. Thus, $C$ has the Hamming distances computed as

- $d_H(C)=3$, if $l=0, p^s+1 \le j \le k \le p^s+2p^{s-1}$ and $p^s+p^{s-1}+1 \le i \le p^s+2p^{s-1}$
- $d_H(C)=4$, if $l=0$, $p^s+1 \le j \le k \le 2p^s$ and $p^s+2p^{s-1}+1 \le i \le 2p^s$

from Theorem 2.12.

**Subcase 4.3** $1 \le j \le p^s$ and $p^s+1 \le k \le i \le 2p^s$.

Then, clearly $\langle (x-\alpha_0)^i (x+\alpha_0)^j (x-\eta\alpha_0)^k \rangle \supseteq \langle u(x-\alpha_0)^{i-p^s}(x-\eta\alpha_0)^{k-p^s} \rangle$. Thus, $C$ has a Hamming distance of 3 from Theorem 2.12.

**Subcase 4.4** $1 \le j \le k \le p^s$ and $p^s+1 \le i \le 2p^s$.

Then, clearly $\langle (x-\alpha_0)^i (x+\alpha_0)^j (x-\eta\alpha_0)^k \rangle \supseteq \langle u(x-\alpha_0)^{i-p^s} \rangle$. Thus, $C$ has a Hamming distance of 2 from Theorem 2.12.

**Case 5** $i \ne 0$, $j \ne 0, k \ne 0$ and $l \ne 0$.

**Subcase 5.1** $1 \le l \le k \le j \le i \le p^s$.

Then, we have $u \in \langle (x-\alpha_0)^i (x+\alpha_0)^j (x-\eta\alpha_0)^k (x+\eta\alpha_0)^l \rangle$ and thus $\langle (x-\alpha_0)^i (x+\alpha_0)^j (x-\eta\alpha_0)^k (x+\eta\alpha_0)^l \rangle$ has a Hamming distance 1.

**Subcase 5.2** $p^s+1 \le l \le k \le j \le i \le 2p^s-1$.

We have, $\langle (x-\alpha_0)^i (x+\alpha_0)^j (x-\eta\alpha_0)^k (x+\eta\alpha_0)^l \rangle = \langle u(x-\alpha_0)^{i-p^s}(x+\alpha_0)^{j-p^s}(x-\eta\alpha_0)^{k-p^s}(x+\eta\alpha_0)^{l-p^s} \rangle$. So, the code $\langle (x-\alpha_0)^i (x+\alpha_0)^j (x-\eta\alpha_0)^k (x+\eta\alpha_0)^l \rangle$ in $R_{\alpha+u\beta}$ has the same Hamming distances as the code $\langle (x-\alpha_0)^{i-p^s}(x+\alpha_0)^{j-p^s}(x-\eta\alpha_0)^{k-p^s}(x+\eta\alpha_0)^{l-p^s} \rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha \rangle}$, multiplied by $u$. Thus, $C$ has the Hamming distances computed as

$$d_H(C)=\min\{(\rho_0+1)p^{\kappa_0}, 2(\rho_1+1)p^{\kappa_1}, 3(\rho_2+1)p^{\kappa_2}, 4(\rho_3+1)p^{\kappa_3}\}, \text{if } 2p^s-p^{s-\kappa_0}+(\rho_0-1)p^{s-\kappa_0-1}+1 \le i \le 2p^s-p^{s-\kappa_0}+\rho_0 p^{s-\kappa_0-1}, 2p^s-p^{s-\kappa_1}+(\rho_1-1)p^{s-\kappa_1-1}+1 \le j \le 2p^s-p^{s-\kappa_1}+\rho_1 p^{s-\kappa_1-1}, 2p^s-p^{s-\kappa_2}+(\rho_2-1)p^{s-\kappa_2-1}+1 \le k \le 2p^s-p^{s-\kappa_2}+\rho_2 p^{s-\kappa_2-1}, 2p^s-p^{s-\kappa_3}+(\rho_3-1)p^{s-\kappa_3-1}+1 \le l \le 2p^s-p^{s-\kappa_3}+\rho_3 p^{s-\kappa_3-1}$$

**Subcase 5.3** $i=2p^s$ and $p^s+1 \le l \le k \le j \le 2p^s-1$.

We have, $\langle (x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle = \langle u(x-\alpha_0)^{p^s}(x+\alpha_0)^{j-p^s}(x-\eta\alpha_0)^{k-p^s}(x+\eta\alpha_0)^{l-p^s}\rangle$. So, the code $\langle (x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle$ in $R_{\alpha+u\beta}$ has the same Hamming distances as the code $\langle (x-\alpha_0)^{p^s}(x+\alpha_0)^{j-p^s}(x-\eta\alpha_0)^{k-p^s}(x+\eta\alpha_0)^{l-p^s}\rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha\rangle}$, multiplied by $u$. Thus, $C$ has the Hamming distances computed as

$$d_H(C) = \min\{2(\rho_1+1)p^{\kappa_1}, 3(\rho_2+1)p^{\kappa_2}, 4(\rho_3+1)p^{\kappa_3}\}, \text{if } i = 2p^s, 2p^s - p^{s-\kappa_1} + (\rho_1-1)p^{s-\kappa_1-1}+1 \le j \le 2p^s - p^{s-\kappa_1} + \rho_1 p^{s-\kappa_1-1}, 2p^s - p^{s-\kappa_2} + (\rho_2-1)p^{s-\kappa_2-1}+1 \le k \le 2p^s - p^{s-\kappa_2} + \rho_2 p^{s-\kappa_2-1}, 2p^s - p^{s-\kappa_3} + (\rho_3-1)p^{s-\kappa_3-1}+1 \le l \le 2p^s - p^{s-\kappa_3} + \rho_3 p^{s-\kappa_3-1}$$

**Subcase 5.4** $i = j = 2p^s$ and $p^s+1 \le l \le k \le 2p^s-1$.

We have, $\langle (x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle = \langle u(x-\alpha_0)^{p^s}(x+\alpha_0)^{p^s}(x-\eta\alpha_0)^{k-p^s}(x+\eta\alpha_0)^{l-p^s}\rangle$. So, the code $\langle (x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle$ in $R_{\alpha+u\beta}$ has the same Hamming distances as the code $\langle (x-\alpha_0)^{p^s}(x+\alpha_0)^{p^s}(x-\eta\alpha_0)^{k-p^s}(x+\eta\alpha_0)^{l-p^s}\rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha\rangle}$, multiplied by $u$. Thus, $C$ has the Hamming distances computed as

$$d_H(C) = \min\{3(\rho_2+1)p^{\kappa_2}, 4(\rho_3+1)p^{\kappa_3}\}, \text{if } i = j = 2p^s, 2p^s - p^{s-\kappa_2} + (\rho_2-1)p^{s-\kappa_2-1}+1 \le k \le 2p^s - p^{s-\kappa_2} + \rho_2 p^{s-\kappa_2-1}, 2p^s - p^{s-\kappa_3} + (\rho_3-1)p^{s-\kappa_3-1}+1 \le l \le 2p^s - p^{s-\kappa_3} + \rho_3 p^{s-\kappa_3-1}$$

**Subcase 5.5** $i = j = k = 2p^s$ and $p^s+1 \le l \le 2p^s-1$.

We have, $\langle (x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle = \langle u(x-\alpha_0)^{p^s}(x+\alpha_0)^{p^s}(x-\eta\alpha_0)^{p^s}(x+\eta\alpha_0)^{l-p^s}\rangle$. So, the code $\langle (x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle$ in $R_{\alpha+u\beta}$ has the same Hamming distances as the code $\langle (x-\alpha_0)^{p^s}(x+\alpha_0)^{p^s}(x-\eta\alpha_0)^{p^s}(x+\eta\alpha_0)^{l-p^s}\rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha\rangle}$, multiplied by $u$. Thus, $C$ has the Hamming distances computed as

$d_H(C) = 4(\rho_3+1)p^{\kappa_3}$, if $i = j = k = 2p^s, 2p^s - p^{s-\kappa_3} + (\rho_3-1)p^{s-\kappa_3-1}+1 \le l \le 2p^s - p^{s-\kappa_3} + \rho_3 p^{s-\kappa_3-1}$.

**Subcase 5.6** $1 \le l \le p^s$ and $p^s+1 \le k \le j \le i \le 2p^s$.

We have, $\langle (x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle \supseteq \langle u(x-\alpha_0)^{i-p^s}(x+\alpha_0)^{j-p^s}(x-\eta\alpha_0)^{k-p^s}\rangle$. So, the code $\langle (x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle$ in $R_{\alpha+u\beta}$ has the same Hamming distances as the code $\langle (x-\alpha_0)^{i-p^s}(x+\alpha_0)^{j-p^s}(x-\eta\alpha_0)^{k-p^s}\rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha\rangle}$, multiplied by $u$. Thus, $C$ has the Hamming distances computed as

- $d_H(C) = 3$, if $1 \le l \le p^s, p^s+1 \le j \le k \le p^s+2p^{s-1}$ and $p^s+p^{s-1}+1 \le i \le p^s+2p^{s-1}$
- $d_H(C) = 4$, if $1 \le l \le p^s$,, $p^s+1 \le j \le k \le 2p^s$ and $p^s+2p^{s-1}+1 \le i \le 2p^s$

from Theorem 2.12.

**Subcase 5.7** $1 \le l \le j \le p^s$ and $p^s+1 \le k \le i \le 2p^s$.

We have, $\langle (x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle \supseteq \langle u(x-\alpha_0)^{i-p^s}(x+\eta\alpha_0)^{k-p^s}\rangle$. So, the code $\langle (x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle$ in $R_{\alpha+u\beta}$ has the same Hamming distances as the code $\langle (x-\alpha_0)^{i-p^s}(x+\eta\alpha_0)^{k-p^s}\rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha\rangle}$, multiplied by $u$. Thus, $C$ has the Hamming distances computed as 3 from Theorem 2.12.

**Subcase 5.8** $1 \le l \le j \le k \le p^s$ and $p^s+1 \le i \le 2p^s$.

We have, $\langle (x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle \supseteq \langle u(x-\alpha_0)^{i-p^s}\rangle$. So, the code $\langle (x-\alpha_0)^i(x+\alpha_0)^j(x-\eta\alpha_0)^k(x+\eta\alpha_0)^l\rangle$ in $R_{\alpha+u\beta}$ has the same Hamming distances as the code $\langle (x-\alpha_0)^{i-p^s}\rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha\rangle}$, multiplied by $u$. Thus, $C$ has the Hamming distances computed as 2 from Theorem 2.12.

**Subcase 5.9** $i=j=k=l=2p^s$.

Then, $C$ has Hamming distance of 0.

Combining all the cases we get the Hamming distances of $(\alpha+u\beta)$-constacyclic codes when $l\le k\le j\le i$. ■

**Remark 3.13** Using the same technique as above, it is easy to check that the corresponding cases with $j\le l\le k\le i, j\le k\le l\le i, k\le j\le l\le i, l\le i\le k\le j, i\le k\le l\le j, k\le i\le l\le j, i\le l\le k\le j, l\le i\le j\le k, l\le j\le i\le k, i\le j\le l\le k, i\le k\le j\le l, j\le l\le i\le k, k\le j\le i\le l, k\le i\le j\le l$ and $j\le k\le i\le l$ have the same Hamming distances as case $l\le j\le k\le i$ in the above Theorem.

**Structure IV: When $p^m\equiv 1(\text{mod}4)$ and $\lambda=\alpha+u\beta$ is a square of the form $\lambda=\lambda_0^2$, where $\lambda_0$ is non-square in $R$.**

In this case, there exists $\alpha_0\in\mathbb{F}_{p^m}^*$ such that $\alpha=\alpha_0^{2p^s}$. Obviously, $\alpha_0$ is also non-square. Then, $(x^2+\alpha_0)$ and $(x^2+\alpha_0)$ are irreducible in $\mathbb{F}_{p^m}[x]$. $x^{4p^s}-\alpha$ has the factorization into product of irreducible factors as $x^{4p^s}-\alpha=(x^2-\alpha_0)^{p^s}(x^2+\alpha_0)^{p^s}$.

The ring $R_{\alpha+u\beta}$ is a principal ideal ring whose ideals i.e., $(\alpha+u\beta)$-constacyclic code of length $4p^s$ over $R$ are $C=\langle (x^2-\alpha_0)^i(x^2+\alpha_0)^j\rangle$ , where $0\le i,j\le 2p^s$. Then, we have the following lemma:

**Lemma 3.14** In $R_{\alpha+u\beta}$, $\langle (x^2-\alpha_0)^{p^s}(x^2+\alpha_0)^{p^s}\rangle=\langle u\rangle$. In particular, $(x^2-\alpha_0)(x^2+\alpha_0)$ is nilpotent in $R_{\alpha+u\beta}$ with nilpotency index $2p^s$.

Now, we here consider the Hamming distance of $C$ for the case $0\le j\le i\le 2p^s$. **Theorem 3.15** The $(\alpha+u\beta)$-constacyclic codes of length $4p^s$ over $R$,

$$C=\langle (x^2-\alpha_0)^i(x^2+\alpha_0)^j\rangle$$

for $0\le j\le i\le 2p^s$ have the following Hamming distances

- $d_H(C)=1, \text{if} 0\le j\le i\le p^s$
- $d_H(C)=2, \text{If } 0\le j\le p^s$ and $p^s+1\le i\le 2p^s$
- $d_H(C)=\min\{(\rho_0+1)p^{\kappa_0}, 2(\rho_1+1)p^{\kappa_1}\}, \text{if } 2p^s-p^{s-\kappa_0}+(\rho_0-1)p^{s-\kappa_0-1}+1\le i \le 2p^s-p^{s-\kappa_0}+\rho_0p^{s-\kappa_0-1}, 2p^s-p^{s-\kappa_1}+(\rho_1-1)p^{s-\kappa_1-1}+1\le j \le 2p^s-p^{s-\kappa_1}+\rho_1p^{s-\kappa_1-1}$
- $d_H(C)=2(\rho_1+1)p^{\kappa_1}, \text{if } i=2p^s, 2p^s-p^{s-\kappa_1}+(\rho_1-1)p^{s-\kappa_1-1}+1\le j \le 2p^s-p^{s-\kappa_1}+\rho_1p^{s-\kappa_1-1}$
- $d_H(C)=0, \text{if } i=j=2p^s$

where $1\le\rho_0,\rho_1\le p-1$, $0\le\kappa_1\le\kappa_0\le s-1$.

***Proof.***

**Case 1** $i=j=0$. Then, the code has Hamming distance of 1.

**Case 2** $j=0$ and $i\ne 0$.

**Subcase 2.1** $1\le i\le p^s$.

Then, clearly, we have $u\in\langle (x^2-\alpha_0)^i\rangle$ and thus $\langle (x^2-\alpha_0)^i\rangle$ has a Hamming distance of 1.

**Subcase 2.2** $p^s+1\le i\le 2p^s$.

Then, clearly $\langle (x^2-\alpha_0)^i\rangle \supseteq \langle u(x^2-\alpha_0)^{i-p^s}\rangle$.

So, $\langle (x^2-\alpha_0)^i\rangle$ has the same Hamming distances as the code $\langle (x^2-\alpha_0)^{i-p^s}\rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha\rangle}$, multiplied by $u$. Hence, from Theorem 2.14, we have Hamming distance of 2.

**Case 3** $i\neq 0, j\neq 0$

**Subcase 3.1** $1\le j\le i\le p^s$.

Then, $u\in\langle (x^2-\alpha_0)^i(x^2+\alpha_0)^j\rangle$ and thus $\langle (x^2-\alpha_0)^i(x^2+\alpha_0)^j\rangle$ has a Hamming distance of 1.

**Subcase 3.2** $p^s+1\le j\le i\le 2p^s-1$.

Then $\langle (x^2-\alpha_0)^i(x^2+\alpha_0)^j\rangle=\langle u(x^2-\alpha_0)^{i-p^s}(x^2+\alpha_0)^{j-p^s}\rangle$.

So, the code $\langle (x^2-\alpha_0)^i(x^2+\alpha_0)^j\rangle$ in $R_{\alpha+u\beta}$ has the Hamming distances same as the code $\langle (x^2-\alpha_0)^{i-p^s}(x^2+\alpha_0)^{j-p^s}\rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha\rangle}$, multiplied by $u$. Thus, from Theorem 2.14, the Hamming distances computed as

$$\begin{aligned}d_H(C)=\min\{(\rho_0+1)p^{\kappa_0},2(\rho_1+1)p^{\kappa_1}\}, &\text{if } 2p^s-p^{s-\kappa_0}+(\rho_0-1)p^{s-\kappa_0-1}+1\le i\\ &\le 2p^s-p^{s-\kappa_0}+\rho_0p^{s-\kappa_0-1}, 2p^s-p^{s-\kappa_1}+(\rho_1-1)p^{s-\kappa_1-1}+1\le j\\ &\le 2p^s-p^{s-\kappa_1}+\rho_1p^{s-\kappa_1-1}\end{aligned}$$

**Subcase 3.3** $i=2p^s, p^s+1\le j\le 2p^s-1$.

Then, $\langle (x^2-\alpha_0)^i(x^2+\alpha_0)^j\rangle=\langle u(x^2-\alpha_0)^{p^s}(x^2+\alpha_0)^{j-p^s}\rangle$.

So, the code $\langle (x^2-\alpha_0)^i(x^2+\alpha_0)^j\rangle$ in $R_{\alpha+u\beta}$ has the same Hamming distance as the code $\langle (x^2-\alpha_0)^{p^s}(x^2+\alpha_0)^{j-p^s}\rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha\rangle}$, multiplied by $u$. Thus, from Theorem 2.14, the Hamming distances computed as

$$\begin{aligned}d_H(C)=2(\rho_1+1)p^{\kappa_1}, &\text{if } i=2p^s, 2p^s-p^{s-\kappa_1}+(\rho_1-1)p^{s-\kappa_1-1}+1\le j\\ &\le 2p^s-p^{s-\kappa_1}+\rho_1p^{s-\kappa_1-1}\end{aligned}$$

**Subcase 3.4** $1\le j\le p^s$ and $p^s+1\le i\le 2p^s$.

Then, $\langle (x^2-\alpha_0)^i(x^2+\alpha_0)^j\rangle\supseteq\langle u(x^2-\alpha_0)^{i-p^s}\rangle$.

So, $\langle (x^2-\alpha_0)^i(x^2+\alpha_0)^j\rangle$ has the same Hamming distances as the code $\langle (x^2-\alpha_0)^{i-p^s}\rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\alpha\rangle}$, multiplied by $u$. Thus $\langle (x^2-\alpha_0)^i(x^2+\alpha_0)^j\rangle$ has a Hamming distance of 2. **Subcase 3.5** $i=j=2p^s$

Then, $\langle (x^2-\alpha_0)^i(x^2+\alpha_0)^j\rangle$ has Hamming distance 0.

Combining all the cases we get the Hamming distances of $(\alpha+u\beta)$-constacyclic codes when $j\le i$. ■

**Remark 3.16** Using the same technique as above, it is easy to check that the corresponding case with $i\le j$ has the same Hamming distances as $j\le i$ in the above Theorem.

**Structure V: $p^m\equiv 1(\mathrm{mod}\,4)$ and $\lambda=\alpha+u\beta$ is non-square unit in $R$.**

Then, there exists $\alpha_0\in\mathbb{F}_{p^m}^*$ such that $\alpha=\alpha_0^{p^s}$. Also, $(x^4-\alpha_0)$ is irreducible in $\mathbb{F}_{p^m}[x]$, and so, $x^{4p^s}-\alpha=(x^4-\alpha_0)^{p^s}$.

Now, consider the ring $R_{\alpha+u\beta}$, whose ideals are of the form $C=\langle (x^4-\alpha_0)^i\rangle$, where $0\le i\le 2p^s$. Equivalently, $C=\langle (x^4-\alpha_0)^i\rangle$, $0\le i\le 2p^s$ are $(\alpha+u\beta)$-constacyclic codes of length $4p^s$ over $R$. Then, the following lemma follows:

**Lemma 3.17** In $R_{\alpha+u\beta}$, $\langle (x^4-\alpha_0)^{p^s}\rangle=\langle u\rangle$. In particular, $(x^4-\alpha_0)$ is nilpotent in $R_{\alpha+u\beta}$ with nilpotency index $2p^s$.

The Hamming distance distribution $d_H(C)$ of $(\alpha+u\beta)$-constacyclic codes of length $4p^s$ over $R$ is completely determined as follows:

**Theorem 3.18** Let $C = \langle (x^4-\alpha_0)^i\rangle$, for $i \in \{0,1,\dots,2p^s\}$, then the Hamming distance $d_H(C)$ is completely determined by

- $d_H(C) = 1$, if $0 \leq i \leq p^s$
- $d_H(C) = (\rho_0+1)p^{\kappa_0}$, if $2p^s - p^{s-\kappa_0} + (\rho_0-1)p^{s-\kappa_0-1} + 1 \leq i \leq 2p^s - p^{s-\kappa_0} + \rho_0 p^{s-\kappa_0-1}$
- $d_H(C) = 0$, if $i = 2p^s$

***Proof.***

**Case 1** $1 \leq i \leq p^s$. Then, $u \in (\langle x^4-\alpha_0)^i\rangle$ and thus $d_H(C) = 1$.

**Case 2** $p^s+1 \leq i \leq 2p^s-1$. Then,$\langle (x^4-\alpha_0)^i\rangle = \langle u(x^4-\alpha_0)^{i-p^s}\rangle$, which means that the codewords of the code $\langle (x^4-\alpha_0)^i\rangle$ in $R_{\alpha,\beta}$ are precisely the codewords of the code $\langle (x^4-\alpha_0)^{i-p^s}\rangle$ in $\frac{\mathbb{F}_{p^m}}{\langle x^{4p^s}-\alpha\rangle}$, multiplied by u, which have exactly same Hamming weights. Moreover, the code $\langle (x^4-\alpha_0)\rangle^{i-p^s}$ of length $4p^s$ have Hamming distances determined as Theorem 2.16. ■

## 4 Examples

In this section, we provide some examples of $(\alpha+u\beta)$-constacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$, where $\alpha,\beta \in \mathbb{F}_{p^m}^*$ with new and good parameters from existing one according to [20].

**Table II. Examples of $\lambda$-constacyclic codes over $\mathbb{F}_5 + u\mathbb{F}_5$**

| $n$ | $\lambda$ | Generator $< g(x) >$ | $[n, M, d_H]$ |
|---|---|---|---|
| 20 | $1+3u$ | $\langle (x-1)^6(x-2)(x-3)(x-4)^0\rangle$ | $[20, 5^{32}, 2]^*$ |
| 20 | $2+u$ | $\langle (x^4-2)^8\rangle$ | $[20, 5^8, 4]^*$ |
| 20 | $4+2u$ | $\langle (x^2-2)^{10}(x^2-3)^9\rangle$ | $[20, 5^2, 10]^*$ |
| 100 | $3+2u$ | $\langle (x^4-3)^2\rangle$ | $[100, 5^{384}, 1]^*$ |
| 100 | $4+u$ | $\langle (x^2-2)^0(x^2-3)^{10}\rangle$ | $[100, 5^{360}, 2]^*$ |

**Table III. Examples of $\lambda$-constacyclic codes over $\mathbb{F}_{3^2} + u\mathbb{F}_{3^2}$**

| $n$ | $\lambda$ | Generator $< g(x) >$ | $[n, M, d_H]$ |
|---|---|---|---|
| 12 | $1+u$ | $\langle (x-1)^6(x+1)^4(x+\omega^2)^2(x+\omega^6)^5\rangle$ | $[12, 3^{15}, 4]^*$ |
| 12 | $\omega^2+u$ | $\langle (x^2+\omega^3)^0(x^2+\omega^7)^4\rangle$ | $[12, 3^{16}, 2]^*$ |
| 12 | $\omega+2u$ | $\langle (x^4+\omega^7)^5\rangle$ | $[12, 3^4, 3]^*$ |
| 36 | $1+2u$ | $\langle (x-1)^{18}(x+1)^{18}(x+\omega^2)^{17}(x+\omega^6)^{18}\rangle$ | $[36, 3^2, 36]^*$ |
| 36 | $2+2u$ | $\langle (x+\omega)(x+\omega^3)(x+\omega^5)(x+\omega^7)\rangle$ | $[36, 3^{136}, 1]^*$ |

**Table IV. Examples of $\lambda$-constacyclic codes over $\mathbb{F}_7 + u\mathbb{F}_7$**

| $n$ | $\lambda$ | Generator $< g(x) >$ | $[n, M, d_H]$ |
|---|---|---|---|
| 28 | $1+3u$ | $\langle (x-1)(x+1)(x^2+1)\rangle$ | $[28, 7^{52}, 1]^*$ |
| 28 | $2+u$ | $\langle (x-2)^{11}(x+2)^8(x^2+4)^6\rangle$ | $[28, 7^{25}, 2]^*$ |
| 28 | $4+2u$ | $\langle (x-3)^{13}(x+3)^{13}(x^2+2)^{13}\rangle$ | $[28, 7^4, 7]^*$ |
| 28 | $5+2u$ | $\langle (x^2+x+4)^9(x^2-x+4)^2\rangle$ | $[28, 7^{34}, 3]^*$ |
| 28 | $3+u$ | $\langle (x^2+2x+2)^{11}(x^2-2x+2)^8\rangle$ | $[28, 7^{18}, 5]^*$ |

**Table V. Examples of $\lambda$-constacyclic codes over $\mathbb{F}_{11}+u\mathbb{F}_{11}$**

| $n$ | $\lambda$ | Generator $< g(x) >$ | $[n, M, d_H]$ |
|---|---|---|---|
| 44 | $1+8u$ | $\langle (x+1^{14})(x+10)(x^2+1)^{12}\rangle$ | $[44,11^{49},4]^*$ |
| 44 | $2+u$ | $\langle (x^2+4x+8)^{22}(x^2+7x+8)^{21}\rangle$ | $[44,11^{2},33]^*$ |
| 44 | $6+7u$ | $\langle (x^2+5x+7)^{16}(x^2+6x+7)^{22}\rangle$ | $[44,11^{12},18]^*$ |
| 44 | $5+5u$ | $\langle (x+2)^{22}(x+9)^{21}(x^2+4)^{22}\rangle$ | $[44,11^{1},44]^*$ |
| 44 | $10+3u$ | $\langle (x^2+3x+10)(x^2+8x+10)^{13}\rangle$ | $[44,11^{60},3]^*$ |

## 5 Maximum Distance Separable Codes

In [21], Norton et al. discussed the Singleton bound for finite chain ring $R$ with respect to the Hamming distance $d_H(C)$ and is given as $|C| \le |R|^{n-d_H(C)+1}$. Maximum Distance Separable (MDS) codes are classified as an important class of linear codes that meet the Singleton bound. They have high error correction capability as compared to non MDS codes.

**Theorem 5.1** (Singleton Bound) [21] Let $C$ be a linear code of length $n$ over $R$ with Hamming distance $d_H(C)$. Then, the Singleton bound is given by $|C| \le p^{2m(n-d_H(C)+1)}$. **Definition 1** Let $C$ be a linear code of length $n$ over $R$ with Hamming distance $d_H(C)$. Then, $C$ is said to be a maximum distance separable (MDS) code if it attains the Singleton bound.

In this section, we explore all MDS $(\alpha+u\beta)$-constacyclic codes of length $4p^s$.

**Theorem 5.2** Let $C = \langle (x-\alpha_0)^i (x+\alpha_0)^j (x^2+{\alpha_0}^2)^k\rangle$ be a $(\alpha+u\beta)$-constacyclic code of length $4p^s$ over $R$. Then, the only MDS code for the Hamming distance is the ambient ring $R_{\alpha+u\beta} = \frac{R[x]}{\langle x^{4p^s}-(\alpha+u\beta)\rangle}$ itself.

***Proof.*** Let $C$ be a $(\alpha+u\beta)$-costacyclic code of length $4p^s$ over $R$. From Theorem 2.3, we have $|C| = p^{m(8p^s-i-j-2k)}$.

Then, $C$ is MDS if and only if $|C| = p^{2m(4p^s-d_H(C)+1)}$ i.e., $p^{m(8p^s-i-j-2k)} = p^{2m(4p^s-d_H(C)+1)}$ i.e., $8p^s-i-j-2k = 8p^s-2d_H(C)+2$ i.e., $i+j+2k = 2d_H(C)-2$ . Now, we consider the conditions for the equations hold from the following cases.

**Case 1** $0 \le i,j,k \le p^s$. Then, $d_H(C)=1$, obviously we have $i+j+2k = 2d_H(C)-2$ if and only if $i=j=k=0$. Hence, $C=\langle 1\rangle$ is a MDS $(\alpha+u\beta)$-constacyclic code of length $4p^s$ over $R$.

**Case 2** $0 \le k \le p^s$,$0 \le j \le 2p^s$ and $p^s+1 \le i \le 2p^s$, or $0 \le j \le p^s$, $p^s+1 \le k \le p^s+p^{s-1}$ and $p^s+1 \le i \le p^s+p^{s-1}$ . Then, $d_H(C)=2$ . Also, we have $i+j+2k \ge p^s+1 > 2\cdot 2-2 = 2d_H(C)-2$. Thus, there is no MDS code.

**Case 3** $0 \le j \le p^s$, $p^s+1 \le k \le p^s+2p^{s-1}$ and $p^s+p^{s-1}+1 \le i \le p^s+2p^{s-1}$. Then, $d_H(C) = 3$ and $i+j+2k \ge 3p^s+p^{s-1}+3 > 2\cdot 3-2 = 2d_H(C)-2$. Therefore, no MDS code exists.

**Case 4** $0 \le j \le p^s$, $p^s+1 \le k \le 2p^s$ and $p^s+2p^{s-1}+1 \le i \le 2p^s$. Then, $d_H(C)=4$ and we have $i+j+2k \ge 3p^s+2p^{s-1}+3 > 2\cdot 4-2 = 2d_H(C)-2$. Thus, no MDS code exists.

**Case 5** $2p^s-p^{s-\kappa_0}+(\rho_0-1)p^{s-\kappa_0-1}+1 \le i \le 2p^s-p^{s-\kappa_0}+\rho_0 p^{s-\kappa_0-1}$ , $2p^s-p^{s-\kappa_1}+(\rho_1-1)p^{s-\kappa_1-1}+1 \le k \le 2p^s-p^{s-\kappa_1}+\rho_1 p^{s-\kappa_1-1}$ and $2p^s-p^{s-\kappa_2}+(\rho_2-1)p^{s-\kappa_2-1}+1 \le j \le 2p^s-p^{s-\kappa_2}+\rho_2 p^{s-\kappa_2-1}$. Then, $d_H(C) = \min\{(\rho_0+1)p^{\kappa_0}, 2(\rho_1+1)p^{\kappa_1}, 4(\rho_2+1)p^{\kappa_2}\}$, and

$$\begin{aligned} i+j+2k \geq\ & 8p^s - p^{s-\kappa_0} - 2p^{s-\kappa_1} - p^{s-\kappa_2} + (\rho_0-1)p^{s-\kappa_0-1} + 2(\rho_1-1)p^{s-\kappa_1-1} + (\rho_2 - 1)p^{s-\kappa_2-1} + 4 \\ =\ & 4(2p^s - p^{s-\kappa_0} + (\rho_0-1)p^{s-\kappa_0-1} + 1) \\ & (\text{equality when } \kappa_0=\kappa_1=\kappa_2 \text{ and } \rho_0=\rho_1=\rho_2) \\ =\ & 4(2p^{s-\kappa_0}(p^{\kappa_0}-1) + p^{s-\kappa_0} + (\rho_0-1)p^{s-\kappa_0-1} + 1) \\ \geq\ & 4(2p(p^{\kappa_0}-1) + p + (\rho_0-1) + 1)(\text{equality when } \kappa_0 = s-1) \\ \geq\ & 4(2(\rho_0+1)p^{\kappa_0} - 1)(\text{equality when } \rho_0 = p-1) \\ =\ & 2\cdot(\rho_0+1)p^{\kappa_0} - 2 + (6(\rho_0+1)p^{\kappa_0} - 2) \\ >\ & 2\cdot(\rho_0+1)p^{\kappa_0} - 2 \\ >\ & 2\cdot\min\{(\rho_0+1)p^{\kappa_0}, 2(\rho_1+1)p^{\kappa_1}, 4(\rho_2+1)p^{\kappa_2}\} - 2 \\ =\ & 2d_H(C) - 2 \end{aligned}$$

Therefore, there is no MDS code.

**Case 6** $i=2p^s$, $2p^s - p^{s-\kappa_1} + (\rho_1-1)p^{s-\kappa_1-1} + 1 \leq k \leq 2p^s - p^{s-\kappa_1} + \rho_1 p^{s-\kappa_1-1}$ and $2p^s - p^{s-\kappa_2} + (\rho_2-1)p^{s-\kappa_2-1} + 1 \leq j \leq 2p^s - p^{s-\kappa_2} + \rho_2 p^{s-\kappa_2-1}$ . Then, $d_H(C) = \min\{2(\rho_1+1)p^{\kappa_1}, 4(\rho_2+1)p^{\kappa_2}\}$, and

$$\begin{aligned} i+j+2k \geq\ & 8p^s - 2p^{s-\kappa_1} - p^{s-\kappa_2} + 2(\rho_1-1)p^{s-\kappa_1-1} + (\rho_2-1)p^{s-\kappa_2-1} + 3 \\ =\ & 5p^s + 3(p^s - p^{s-\kappa_1} + (\rho_1-1)p^{s-\kappa_1-1} + 1) \\ & (\text{equality when } \kappa_1=\kappa_2 \text{ and } \rho_1=\rho_2) \\ =\ & 5p^s + 3(p^{s-\kappa_1}(p^{\kappa_1}-1) + (\rho_1-1)p^{s-\kappa_1-1} + 1) \\ \geq\ & 5p^{\kappa_1+1} + 3(p(p^{\kappa_1}-1) + (\rho_1-1) + 1)(\text{equality when } \kappa_1 = s-1) \\ \geq\ & 5(\rho_1+1)p^{\kappa_1} + 3((\rho_1+1)p^{\kappa_1} - 1)(\text{equality when } \rho_1 = p-1) \\ =\ & 2\cdot 2(\rho_1+1) - 2 + (4(\rho_1+1)p^{\kappa_1} - 1) \\ >\ & 2\cdot 2(\rho_1+1)p^{\kappa_1} - 2 \\ >\ & 2\cdot\min\{2(\rho_1+1)p^{\kappa_1}, 4(\rho_2+1)p^{\kappa_2}\} - 2 \\ =\ & 2d_H(C) - 2 \end{aligned}$$

Therefore, there is no MDS code.

**Case 7** $i=2p^s$, $k=2p^s$ and $2p^s - p^{s-\kappa_2} + (\rho_2-1)p^{s-\kappa_2-1} + 1 \leq j \leq 2p^s - p^{s-\kappa_2} + \rho_2 p^{s-\kappa_2-1}$. Then, $d_H(C) = 4(\rho_2+1)p^{\kappa_2}$, and

$$\begin{aligned} i+j+2k \geq\ & 8p^s - p^{s-\kappa_2} + (\rho_2-1)p^{s-\kappa_2-1} + 1 \\ =\ & 7p^s + (p^s - p^{s-\kappa_2} + (\rho_2-1)p^{s-\kappa_2-1} + 1) \\ =\ & 7p^s + (p^{s-\kappa_2}(p^{\kappa_2}-1) + (\rho_2-1)p^{s-\kappa_2-1} + 1) \\ \geq\ & 7p^{\kappa_2+1} + (p(p^{\kappa_2}-1) + (\rho_2-1) + 1)(\text{equalitywhen } \kappa_2 = s-1) \\ \geq\ & 8(\rho_2+1)p^{\kappa_2} - 1(\text{equalitywhen } \rho_2 = p-1) \\ >\ & 2\cdot 4(\rho_2+1)p^{\kappa_2} - 2 \\ =\ & 2d_H(C) - 2 \end{aligned}$$

Therefore, there is no MDS code.

**Case 8** $i=j=k=2p^s$. Then, $d_H(C)=0$, obviously $i+j+2k > 2d_H(C)-2$. Combining all the cases, the result follows. ■

**Theorem 5.3** Let $C = \langle (x-\alpha_0)^i (x+\alpha_0)^j (x-\eta\alpha_0)^k (x+\eta\alpha_0)^l \rangle$ be a $(\alpha+u\beta)$-constacyclic code of length $4p^s$ over $R$. Then, the only MDS code for the Hamming distance is the ambient ring $R_{\alpha+u\beta} = \frac{R[x]}{\langle x^{4p^s} - (\alpha+u\beta)\rangle}$ itself.

***Proof.*** Let $C$ be a $(\alpha+u\beta)$-costacyclic code of length $4p^s$ over $R$. From Theorem 2.3, we have $|C| =$

$p^{m(8p^s-i-j-k-l)}$.

Then, $C$ is MDS if and only if $|C| = p^{2m(4p^s-d_H(C)+1)}$ i.e.,$p^{m(8p^s-i-j-k-l)} = p^{2m(4p^s-d_H(C)+1)}$ i.e., $8p^s - i - j - k - l = 8p^s - 2d_H(C) + 2$ i.e., $i + j + k + l = 2d_H(C) - 2$ . Now, by proceeding similar way as Theorem 5.2, we get the result. ■

**Theorem 5.4** Let $C = \langle (x^2 + 2\delta x + 2\delta^2)^i (x^2 - 2\delta x + 2\delta^2)^j \rangle$ be a $(\alpha + u\beta)$-constacyclic code of length $4p^s$ over $R$. Then, the only MDS code for the Hamming distance is the ambient ring $R_{\alpha+u\beta} = \frac{R[x]}{\langle x^{4p^s}-(\alpha+u\beta)\rangle}$ itself.

*Proof.* Let $C$ be a $(\alpha + u\beta)$-costacyclic code of length $4p^s$ over $R$. From Theorem 2.3, we have $|C| = p^{m(8p^s-2i-2j)}$.

Then, $C$ is MDS if and only if $|C| = p^{2m(4p^s-d_H(C)+1)}$ i.e., $p^{m(8p^s-2i-2j)} = p^{2m(4p^s-d_H(C)+1)}$ i.e., $8p^s - 2i - 2j = 8p^s - 2d_H(C) + 2$ i.e., $2i + 2j = 2d_H(C) - 2$. Now, by proceeding similar way as Theorem 5.2, we get the result.■

**Theorem 5.5** Let $C = \langle (x^2 + 2\delta x + 2\delta^2)^i (x^2 - 2\delta x + 2\delta^2)^j \rangle$ be a $(\alpha + u\beta)$-constacyclic code of length $4p^s$ over $R$. Then, the only MDS code for the Hamming distance is the ambient ring $R_{\alpha+u\beta} = \frac{R[x]}{\langle x^{4p^s}-(\alpha+u\beta)\rangle}$ itself.

***Proof.*** Let $C$ be a $(\alpha + u\beta)$-costacyclic code of length $4p^s$ over $R$. From Theorem 2.3, we have $|C| = p^{m(8p^s-2i-2j)}$.

Then, $C$ is MDS if and only if $|C| = p^{2m(4p^s-d_H(C)+1)}$ i.e., $p^{m(8p^s-2i-2j)} = p^{2m(4p^s-d_H(C)+1)}$ i.e., $8p^s - 2i - 2j = 8p^s - 2d_H(C) + 2$ i.e., $2i + 2j = 2d_H(C) - 2$. Now, by proceeding similar way as Theorem 5.2, we get the result.■

**Theorem 5.6** Let $C = \langle (x^2 - \alpha_0)^i (x^2 + \alpha_0)^j \rangle$ be a $(\alpha + u\beta)$-constacyclic code of length $4p^s$ over $R$. Then, the only MDS code for the Hamming distance is the ambient ring $R_{\alpha+u\beta} = \frac{R[x]}{\langle x^{4p^s}-(\alpha+u\beta)\rangle}$ itself.

***Proof.*** Let $C$ be a $(\alpha + u\beta)$-costacyclic code of length $4p^s$ over $R$. From Theorem 2.3, we have $|C| = p^{m(8p^s-2i-2j)}$.

Then, $C$ is MDS if and only if $|C| = p^{2m(4p^s-d_H(C)+1)}$ i.e., $p^{m(8p^s-2i-2j)} = p^{2m(4p^s-d_H(C)+1)}$ i.e., $8p^s - 2i - 2j = 8p^s - 2d_H(C) + 2$ i.e., $2i + 2j = 2d_H(C) - 2$. Now, by proceeding similar way as Theorem 5.2, we get the result.■

**Theorem 5.7** Let $C = \langle (x^4 - \alpha_0)^i \rangle$ be a $(\alpha + u\beta)$-constacyclic code of length $4p^s$ over $R$. Then, the only MDS code for the Hamming distance is the ambient ring $R_{\alpha+u\beta} = \frac{R[x]}{\langle x^{4p^s}-(\alpha+u\beta)\rangle}$ itself.

***Proof.*** Let $C$ be a $(\alpha + u\beta)$-costacyclic code of length $4p^s$ over $R$. From Theorem 2.3, we have $|C| = p^{m(8p^s-4i)}$.

Then, $C$ is MDS if and only if $|C| = p^{2m(4p^s-d_H(C)+1)}$ i.e., $p^{m(8p^s-4i)} = p^{2m(4p^s-d_H(C)+1)}$ i.e., $8p^s - 4i = 8p^s - 2d_H(C) + 2$ i.e., $4i = 2d_H(C) - 2$ i.e., $2i = d_H(C) - 1$. Now, by proceeding similar way as Theorem 5.2, we get the result.■

## 6 Conclusion

The Hamming distances of constacyclic codes have a significant role in error-correcting coding theory. However, a minimal amount of work has been done on the computation of the Hamming distances as it is generally a very complex task. In this paper, all Hamming distances of repeated-root $(\alpha +$

$u\beta$)-constacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ are determined. Also, we obtained some new parameters of repeated-root constacyclic codes as examples.

**References**

[1] H. Q. Dinh, Constacyclic codes of length $p^s$ over $F_{p^m} + uF_{p^m}$, *J. Algebra* 324 (2010), 940-950.

[2] K. Guenda and T. A. Gulliver, Constacyclic codes of length $mp^s$ over $F_{p^r} + uF_{p^r} \ldots + u^{e-1}F_{p^r}$, *Journal of Algebra and Its Applications*, 14(01) (2015), 1450081. .

[3] S. R. López-Permouth, H. Özadam, F. Özbudak and S. Szabo, Polycyclic codes over Galois rings with applications to repeated-root constacyclic codes, *Finite Fields and Their Applications*, 19(1) (2013), 16-38. .

[4] H. Q. Dinh, B. T. Nguyen, A. K. Singh and S. Sriboonchitta, Hamming and Symbol-Pair Distances of Repeated-Root Constacyclic Codes of Prime Power Lengths Over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$, *IEEE Communications Letters*, 22(12) (2018), 2400-2403.

[5] H. Q. Dinh, X. Wang, H. Liu and S. Sriboonchitta, On the Hamming distances of repeated-root constacyclic codes of length $4p^s$, *Discrete Mathematics*, 342(5)(2019), 1456-1470.

[6] H.Q.Dinh and S. R. López-Permouth, Cyclic and negacyclic codes over finite chain rings, *IEEE Trans. Inform. Theory* $50(2004), 1728 - 1744$.

[7] V. Pless, R. A. Brualdi and W. C. Huffman, Handbook of coding theory, *Elsevier Science Inc*(1998).

[8] H. Q. Dinh, S. Dhompongsa and S. Sriboonchitta, On constacyclic codes of length $4p^s$ over $F_{p^m} + uF_{p^m}$, *Discrete Mathematics*, 340(4)(2017), 832-849.

[9] H. Q. Dinh, A. Sharma, S. Rani and S.Sriboonchitta, Cyclic and negacyclic codes of length $4p^s$ over $F_{p^m} + uF_{p^m}$, *Journal of Algebra and Its Applications*, 17(09)(2018), 1850173.

[10] H. Q.Dinh, B. T. Nguyen, S. Sriboonchitta and T. M. Vo, On $\alpha + u\beta$-constacyclic codes of length $4p^s$ over $F_{p^m} + uF_{p^m}$, *Journal of Algebra and Its Applications*, 18(02)(2019), 1950023.

[11] H. Q. Dinh, B. T. Nguyen, S. Sriboonchitta and T. M. Vo, On a class of constacyclic codes of length $4p^s$ over $F_{p^m} + uF_{p^m}$, *Journal of Algebra and Its Applications*, 18(02)(2019), 1950022.

[12] H. Q. Dinh, On the linear ordering of some classes of negacyclic and cyclic codes and their distance distributions, *Finite Fields and Their Applications*, 14(1)(2008), 22-40.

[13] H. Q. Dinh, Repeated-root constacyclic codes of length $2p^s$, *Finite Fields and Their Applications*, 18(1)(2012), 133-143.

[14] H. Q. Dinh, Structure of repeated-root constacyclic codes of length $3p^s$ and their duals, *Discrete Mathematics*, 313(9)(2013), 983-991.

[15] H. Q. Dinh, On repeated-root constacyclic codes of length $4p^s$, *Asian-European Journal of Mathematics*, 6(02)(2013), 1350020.

[16] H. Õzadam and F. Õzbudak, The Minimum Hamming Distance of Cyclic Codes of Length $2p^s$, in International Symposium on Applied Algebra, Algebraic Algorithms, and Error-Correcting Codes (2009) (pp. 92-100). *Springer, Berlin, Heidelberg*.

[17] L. Liu, L. Li, X. Kai and S. Zhu, Repeated-root constacyclic codes of length $3lp^s$ and their dual codes, *Finite Fields and Their Applications*, 42(2016), 269-295.

[18] G. Castagnoli, J. L. Massey, P. A. Schoeller and N. von Seemann, On repeated-root cyclic codes, *IEEE Transactions on Information Theory*, 37(2)(1991), 337-342.

[19] J. H. van Lint, Repeated-root cyclic codes, *IEEE Transactions on Information Theory*, 37(2)(1991),

343-345.
[20] M. Grassl, *Bounds on the minimum distance of linear codes and quantum codes,* Online available at http://www.codetables.de. Accessed on 2019-07-25.
[21] G. H. Norton and A. Salagean, *On the Hamming distance of linear codes over a finite chain ring* IEEE Transactions on Information Theory **46**, no. 3 (2000): 1060-1067.