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Coefficient Estimates on A New Subclasses of Bi-Univalent Functions Defined Using Opoola Differential Operator and Associated with Quasi-Subordination

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Abstract

In this research paper, we investigate new subclasses $\mathcal{R}_{\Sigma}^{\mu,q,m}(\lambda,\phi,\gamma,\beta,t)$ and $\mathcal{UAR}_{\Sigma}^{\lambda,q,m}(\phi,\gamma,\beta,t)$ of bi-univalent functions defined on the unit disk Δ in the complex plane using Opoola differential operator and quasi-subordination. Further, we find upper bounds of $|a_2|$ and $|a_3|$ for functions in these new subclasses.

Keywords: Analytic function, Bi-univalent function, Quasi-subordination, Subordination, Univalent function.

Mathematics Subject Classification: 30C45, 30C50, 30C75.

1. Introduction

Let $\mathcal{A}$ be the class of functions in the form

$$h(z) = z + \sum_{j=2}^{\infty} a_j z^j \qquad (0.1)$$

which are analytic in the open unit disk $\Delta = \{z: z \in \mathbb{C}, |z| < 1\}$.

Let S denote the class of functions of $\mathcal{A}$ which are univalent in Δ. As each $h \in S$ is univalent in Δ, h^{-1} exists but it may not be defined on entire Δ. Here the Koebe-one-quarter theorem ([5]) ensures that, the image of every $h \in S$ contains a disk of radius $\frac{1}{4}$.Therefore, for any $h \in S$ having Taylor's series expansion mentioned in equation (0.1) has inverse function g which is given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots \qquad (0.2)$$

where $|w| < r_0(h)$ and $r_0(h) \geq \frac{1}{4}$. A function $h \in \mathcal{A}$ is said to be bi-univalent in Δ if both h and h^{-1} are univalent in Δ. The class of all bi-univalent functions defined in Δ is denoted by $\sum$.

Lewin([8]) invested the class $\sum$ of bi-univalent functions and proved that $|a_2| < 1.5$ for the functions in $\sum$. Later, Brannan and Clunie([3]) conjectured that $|a_2| \leq \sqrt{2}$. Also, Netanyahu([11]) proved that $max_{h\in\sum} |a_2| = \frac{4}{3}$. Still the coefficient bounds for $|a_3|, |a_4|, \ldots$ is an open problem.

The study of subclasses of bi-univalent functions was continued by Brannan and Taha([4]) (see also

([19])) by introducing certain subclasses of bi-univalent functions $S^*(\alpha)$ and $K(\alpha)$ $(\alpha \in [0,1))$ of starlike and convex functions of order α respectively. Srivastava et al.([18]) also contributed by introducing certain subclasses of bi-univalent functions and found out some initial coefficient bounds.
Ma and Minda([9]) introduced the classes:

$$S^*(\phi) = \left\{h \in S: \frac{zh'(z)}{h(z)} \prec \phi(z)\right\}$$

and

$$K(\phi) = \left\{h \in S: 1 + \frac{zh''(z)}{h'(z)} \prec \phi(z)\right\},$$

where ϕ be an analytic functions with positive real part in the unit disk Δ, $\phi'(0) > 1$, $\phi(0) = 1$ and maps Δ on to a region which is starlike with respect to 1 and symmetric with respect to the real axis. These classes include several well known subclasses of starlike and convex functions respectively as special cases.
The next important concept is quasi-subordination which was introduced by Robertson([16]) in 1970.
An analytic function h is quasi-subordination to another analytic function ϕ if there are two analytic functions ψ and ω with conditions $w(0) = 0, |\psi(z)| \leq 1$ and $|w(z)| < 1$ such that $h(z) = \phi(w(z))\psi(z)$ and it is denoted by

$$h(z) \prec_q \phi(z); \quad (z \in \Delta).$$

For $\psi(z) = 1$, we get $h(z) \prec \phi(z)$ in Δ (see ([10]) and ([15]) for quasi-subordination in details).
In this investigation, we assume that

$$\psi(z) = b_0 + b_1 z + b_2 z^2 + \cdots, \quad (z \in \Delta, |\psi(z)| \leq 1) \tag{0.3}$$

and $\phi(z)$ is an analytic function in Δ with form:

$$\phi(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \quad (c_1 > 0). \tag{0.4}$$

In [12], Opoola introduced the following differential operator as follows:

$$D^m(\gamma, \beta, t): \mathcal{A} \to \mathcal{A}$$

$$D^0(\gamma, \beta, t)h(z) = h(z),$$
$$D^1(\gamma, \beta, t)h(z) = zD_t h(z) = tzh'(z) - z(\beta - \gamma)t + [1 + (\beta - \gamma - 1)t]h(z),$$
$$D^m(\gamma, \beta, t)h(z) = zD_t(D^{m-1}(\gamma, \beta, t)h(z)), m \in \mathbb{N} \tag{0.5}$$

If $h(z)$ is given by (0.1), then from (0.5), we see that

$$D^m(\gamma, \beta, t)h(z) = z + \sum_{j=2}^{\infty} [1 + (j + \beta - \gamma - 1)t]^m a_j z^j$$

where, $t \geq 0, \gamma \in [0, \beta]$ and $m \in \mathbb{N} \cup \{0\}$.

Remark 0.1 (i) When $\mathrm{t} = \lambda, \beta = \gamma, \mathrm{D}^{\mathrm{n}}(\gamma, \gamma, \lambda)\mathrm{h}(\mathrm{z}) = \mathrm{D}_\lambda^{\mathrm{n}}\mathrm{h}(\mathrm{z})$ is the Al-Oboudi Differential operator in [2].
(ii) When $\mathrm{t} = 1, \beta = \gamma, \mathrm{D}^{\mathrm{n}}(\gamma, \gamma, 1)\mathrm{h}(\mathrm{z}) = \mathrm{D}^{\mathrm{n}}\mathrm{h}(\mathrm{z})$ is the Salagean Differential operator introduced in [17].

We use the following lemma to derive our results.
Lemma 0.2 [14] If $\mathcal{P}$ denotes the family of all analytic functions in Δ with positive real part and $\mathrm{p} \in \mathcal{P}$

with $p(z) = 1 + c_1 z + c_2 z^2 + \cdots (z \in \Delta)$ then $|c_j| \leq 2$ for each j.

2. COEFFICIENT BOUNDS FOR THE FUNCTION CLASS $\mathcal{R}_{\Sigma}^{\mu,q,m}(\lambda,\phi,\gamma,\beta,t)$

Definition 0.3 A function $h \in \Sigma$ given by (0.1) is said to be in class $\mathcal{R}_{\Sigma}^{\mu,q,m}(\lambda,\phi,\gamma,\beta,t)$ if the following quasi-subordination holds:

$$\left[\lambda(D^m(\gamma,\beta,t)h(z))'\left(\frac{D^m(\gamma,\beta,t)h(z)}{z}\right)^{\mu-1} + (1-\lambda)\left(\frac{D^m(\gamma,\beta,t)h(z)}{z}\right)^{\mu} - 1\right] \prec_q (\phi(z)-1) \qquad (z \in \Delta)$$

and

$$\left[\lambda(D^m(\gamma,\beta,t)g(w))'\left(\frac{D^m(\gamma,\beta,t)g(w)}{w}\right)^{\mu-1} + (1-\lambda)\left(\frac{D^m(\gamma,\beta,t)g(w)}{w}\right)^{\mu} - 1\right] \prec_q (\phi(w)-1) \qquad (w \in \Delta)$$

where $\lambda \in [1,\infty)$ and the functions g and ϕ are given by (0.2) and (0.4) respectively.

Note that, for $\mu = 0$ and $m = 0$, we get the class $\mathcal{R}_{\Sigma}^{0,q,0}(\lambda,\phi,\gamma,\beta,t)$ which was introduced and studied by A.B.Patil and U.H. Naik ([13]).

Theorem 0.4 Let h given by (0.1) be in the class $\mathcal{R}_{\Sigma}^{\mu,q,m}(\lambda,\phi,\gamma,\beta,t)$. Then

$$|a_2| \leq min\left\{\sqrt{\frac{2|b_0|(c_1+|c_2-c_1|)}{(\mu+1)(2\lambda+\mu)(1+(1+\beta-\gamma)t)^{2m}}}, \frac{|b_0|c_1}{(\lambda+\mu)(1+(1+\beta-\gamma)t)^m}\right\} \qquad (0.6)$$

and

$$|a_3| \leq min\left\{\frac{|b_0|^2 c_1^2}{(\lambda+\mu)^2(1+(2+\beta-\gamma)t)^m} + \frac{c_1(|b_0|+|b_1|)}{(2\lambda+\mu)(1+(1+\beta-\gamma)t)^m}, \frac{2|b_0|(c_1+|c_2-c_1|)+(\mu+1)|b_1|c_1}{(\mu+1)(2\lambda+\mu)(1+(1+\beta-\gamma)t)^m}\right\}. \qquad (0.7)$$

Proof. Since $h \in \mathcal{R}_{\Sigma}^{\mu,q,m}(\lambda,\phi,\gamma,\beta,t)$, there exist two analytic functions $u,v:\Delta \to \Delta$ with $|u(z)| < 1, |v(w)| < 1, u(0) = v(0) = 0$ and a function ψ defined by (0.3) satisfies:

$$\left[\lambda\left(D^m(\gamma,\beta,t)h(z)\right)'^{\left(\frac{D^m(\gamma,\beta,t)h(z)}{z}\right)^{\mu-1}} + (1-\lambda)\left(\frac{D^m(\gamma,\beta,t)h(z)}{z}\right)^{\mu} - 1\right] = [\phi(u(z))-1]\psi(z) \qquad (0.8)$$

and

$$\left[\lambda\left(D^m(\gamma,\beta,t)g(w)\right)'^{\left(\frac{D^m(\gamma,\beta,t)g(w)}{w}\right)^{\mu-1}} + (1-\lambda)\left(\frac{D^m(\gamma,\beta,t)g(w)}{w}\right)^{\mu} - 1\right] = [\phi(v(w))-1]\psi(w). \qquad (0.9)$$

Consider functions p and q such that

$$p(z) = \frac{1+u(z)}{1-u(z)} = 1 + \sum_{j=1}^{\infty} d_j z^j$$

equivalently

$$u(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[d_1 z + \left(d_2 - \frac{d_1^2}{2}\right)z^2 + \cdots\right] \qquad (0.10)$$

and

$$q(w)=\frac{1+v(w)}{1-v(w)}=1+\sum_{j=1}^{\infty} e_j w^j$$

equivalently

$v(w)=\frac{q(w)-1}{q(w)+1}=\frac{1}{2}\left[e_1 w+\left(e_2-\frac{e_1^2}{2}\right)w^2+\cdots\right].$ (0.11)

Clearly p and q are both analytic in Δ with $p(0)=q(0)=1$ and have their positive real part in Δ. Now using equations (0.10) and (0.11), R.H.S. of equations (0.8) and (0.9) can be expressed as

$[\phi(u(z))-1]\psi(z)=\frac{b_0c_1d_1}{2}z+\left\{\frac{b_0c_1d_2}{2}-\frac{b_0c_1d_1^2}{4}+\frac{b_0c_2d_1^2}{4}+\frac{b_1c_1d_1}{2}\right\}z^2+\cdots$ (0.12)

and

$[\phi(v(w))-1]\psi(w)=\frac{b_0c_1e_1}{2}w+\left\{\frac{b_0c_1e_2}{2}-\frac{b_0c_1e_1^2}{4}+\frac{b_0c_2e_1^2}{4}+\frac{b_1c_1e_1}{2}\right\}w^2+\cdots.$ (0.13)

By considering functions h and g given by equations (0.1) and (0.2), L.H.S. of equations (0.8) and (0.9) can be expressed as

$$\left[\lambda(D^m(\gamma,\beta,t)h(z))'\left(\frac{D^m(\gamma,\beta,t)h(z)}{z}\right)^{\mu-1}+(1-\lambda)\left(\frac{D^m(\gamma,\beta,t)h(z)}{z}\right)^{\mu}-1\right]$$

$$=(\mu+\lambda)(1+(1+\beta-\gamma)t)^m a_2 z$$

$$+\left[(2\lambda+\mu)(1+(2+\beta-\gamma)t)^m a_3+(\mu-1)\left(\lambda+\frac{\mu}{2}\right)a_2^2(1+(1+\beta-\gamma)t)^{2m}\right]z^2+\cdots$$
(0.14)

and

$$\left[\lambda(D^m(\gamma,\beta,t)g(w))'\left(\frac{D^m(\gamma,\beta,t)g(w)}{w}\right)^{\mu-1}+(1-\lambda)\left(\frac{D^m(\gamma,\beta,t)g(w)}{w}\right)^{\mu}-1\right]$$
$$=-(\lambda+\mu)(1+(1+\beta-\gamma)t)^m a_2 w$$
$$+\left[-(2\lambda+\mu)(1+(2+\beta-\gamma)t)^m a_3+(3+\mu)\left(\lambda+\frac{\mu}{2}\right)a_2^2(1+(1+\beta-\gamma)t)^{2m}\right]w^2+\cdots.$$
(0.15)

Using equations (0.12), (0.13), (0.14) and (0.15) and equating coefficients of like powers of z and w (only first two terms), we get

$(1+(1+\beta-\gamma)t)^m(\mu+\lambda)a_2=\frac{b_0c_1d_1}{2},$ (0.16)

$(2\lambda+\mu)(1+(2+\beta-\gamma)t)^m a_3+(\mu-1)\left(\lambda+\frac{\mu}{2}\right)(1+(1+\beta-\gamma)t)^{2m}a_2^2=\frac{b_0c_1d_2}{2}-\frac{b_0c_1d_1^2}{4}+\frac{b_0c_2d_1^2}{4}+\frac{b_1c_1d_1}{2},$ (0.17)

$-(\lambda+\mu)(1+(1+\beta-\gamma)t)^m a_2=\frac{b_0c_1e_1}{2}$ (0.18)

and

$-(2\lambda+\mu)(1+(2+\beta-\gamma)t)^m a_3+(3+\mu)\left(\lambda+\frac{\mu}{2}\right)(1+(1+\beta-\gamma)t)^{2m}a_2^2=\frac{b_0c_1e_2}{2}-\frac{b_0c_1e_1^2}{4}+\frac{b_0c_2e_1^2}{4}+\frac{b_1c_1e_1}{2}.$ (0.19)

From equations (0.16) and (0.18), we get

$$d_1 = -e_1 \tag{0.20}$$

and

$$8(\lambda+\mu)^2(1+(1+\beta-\gamma)t)^{2m}a_2^2 = b_0^2c_1^2(d_1^2+e_1^2) \tag{0.21}$$

By adding (0.17) and (0.19) in light of (0.20), we get

$$2(1+(1+\beta-\gamma)t)^{2m}(\mu+1)(2\lambda+\mu)a_2^2 = b_0c_1(d_2+e_2)+b_0d_1^2(c_2-c_1). \tag{0.22}$$

By applying lemma (0.2) to equations (0.21) and (0.22), we get the desire result (0.6).
By subtracting (0.19) from (0.17) in light of (0.20), we get

$$a_3 = \frac{(1+(1+\beta-\gamma)t)^{2m}}{(1+(2+\beta-\gamma)t)^m}a_2^2 + \frac{b_0c_1(d_2-e_2)+2b_1c_1d_1}{4(1+(2+\beta-\gamma)t)^m(2\lambda+\mu)}. \tag{0.23}$$

Using equations (0.21) and (0.23), we get

$$a_3 = \frac{b_0^2c_1^2(d_1^2+e_1^2)}{8(1+(2+\beta-\gamma)t)^m(\lambda+\mu)^2} + \frac{b_0c_1(d_2-e_2)+2b_1c_1d_1}{4(1+(2+\beta-\gamma)t)^m(2\lambda+\mu)}. \tag{0.24}$$

Using equations (0.22) and (0.23), we get

$$a_3 = \frac{b_0c_1(d_2+e_2)+b_0d_1^2(c_2-c_1)}{2(1+(2+\beta-\gamma)t)^m(\mu+1)(2\lambda+\mu)} + \frac{b_0c_1(d_2-e_2)+2b_1c_1d_1}{4(1+(2+\beta-\gamma)t)^m(2\lambda+\mu)}. \tag{0.25}$$

By applying lemma (0.2) to equations (0.24) and (0.25), we get the desire result (0.7). This completes the proof of Theorem (0.4).
We observed that, by setting $\mu = 1$ and $m = 0$ in above theorem, we get the result obtained by Amol Patil and Uday Naik ([13]) as follows:

Corollary 0.5 Let $h(z)$ given by (0.1) be in class $\mathcal{R}_{\Sigma}^{1,q,0}(\lambda,\phi)$. Then

$$|a_2| \le min\left\{\frac{|b_0|c_1}{(\lambda+1)}, \sqrt{\frac{2|b_0|(c_1+|c_2-c_1|)}{2\lambda+1}}\right\}$$

and

$$|a_3| \le min\left\{\frac{|b_0|^2c_1^2}{(\lambda+1)^2}+\frac{(|b_0|+|b_1|)c_1}{2\lambda+1}, \frac{|b_0|(c_1+|c_2-c_1|)+|b_1|c_1}{2\lambda+1}\right\}.$$

By setting $\psi(z) = 1$ in corollary 0.5, result of quasi-subordination converts in to following result of subordination.

Corollary 0.6 Let the function $h(z)$ given by (0.1) be in the class $\mathcal{R}_{\Sigma}(\lambda,\phi)$. Then

$$|a_2| \le min\left\{\frac{c_1}{(\lambda+1)}, \sqrt{\frac{c_1+|c_2-c_1|}{2\lambda+1}}\right\}$$

and

$$|a_3| \le min\left\{\frac{c_1^2}{(\lambda+1)^2}+\frac{c_1}{2\lambda+1}, \frac{2c_1+|c_2-c_1|}{2\lambda+1}\right\}.$$

By setting $\lambda = 1$ in corollary (0.6), we get the following corollary.

Corollary 0.7 Let the function h(z) given by (0.1) be in the class $\mathcal{R}_{\Sigma}(\phi)$. Then

$$|a_2| \leq min\left\{\frac{c_1}{2}, \sqrt{\frac{c_1+|c_2-c_1|}{3}}\right\}$$

and

$$|a_3| \leq min\left\{\frac{c_1^2}{4}+\frac{c_1}{3}, \frac{2c_1+|c_2-c_1|}{3}\right\}.$$

Remark 0.8 Corollaries (0.6) and (0.7) are the improvements of the estimates obtained in Theorem 2.1 given by Kumar et al. ([7]) and Theorem 2.1 given by Ali et al. ([1]), respectively.

Remark 0.9 If we set

$$\phi(z) = \frac{1+(1-2\beta)z}{1-z} = 1+2(1-\beta)z+2(1-\beta)z^2+\cdots; \quad (\beta \in [0,1))$$

in corollaries (0.6) and (0.7) then we get the improvements of the estimates obtained in Theorem 3.2 given by Fransin and Aouf ([6]) and Theorem 2 given by Srivastava et al. ([18]), respectively.

3. COEFFICIENT ESTIMATES FOR THE FUNCTION CLASS $\mathcal{UAR}_{\Sigma}^{\lambda,q,m}(\phi,\gamma,\beta,t)$

Definition 0.10 A function $h \in \Sigma$ given by (0.1) is said to be in the class $\mathcal{UAR}_{\Sigma}^{\lambda}(\phi)$ if the following quasi-subordination holds:

$$\left(\frac{z(D^m(\gamma,\beta,t)f(z))'}{D^m(\gamma,\beta,t)f(z)}\right)^{\lambda}\left(1+\frac{z(D^m(\gamma,\beta,t)f(z))''}{(D^m(\gamma,\beta,t)f(z))'}\right)^{1-\lambda} - 1 \prec_q \phi(z)-1 \quad (z \in \Delta)$$

and

$$\left(\frac{z(D^m(\gamma,\beta,t)g(w))'}{D^m(\gamma,\beta,t)g(w)}\right)^{\lambda}\left(1+\frac{w(D^m(\gamma,\beta,t)g(w))''}{(D^m(\gamma,\beta,t)g(w))'}\right)^{1-\lambda} - 1 \prec_q \phi(w)-1 \quad (w \in \Delta)$$

where g and ϕ are the functions given by (0.2) and (0.4) and $\lambda \geq 0$.

Theorem 0.11 Let h(z) given by (0.1) be in the class $\mathcal{UAR}_{\Sigma}^{\lambda}(\phi)$. Then

$$|a_2| \leq min\left\{\frac{|b_0|c_1}{|2-\lambda|(1+(1+\beta-\gamma)t)^m}, \sqrt{\frac{2|b_0|(c_1+|c_2-c_1|)}{|\lambda^2-3\lambda+4|(1+(1+\beta-\gamma)t)^{2m}}}\right\} \quad (0.26)$$

and

$$|a_3| \leq \frac{(|b_0|+|b_1|)c_1}{2|3-2\lambda|(1+(2+\beta-\gamma)t)^m} + min\left\{\frac{|b_0|^2c_1^2}{(2-\lambda)^2(1+(2+\beta-\gamma)t)^m}, \frac{2|b_2|(c_1+|c_2-c_1|)}{|\lambda^2-3\lambda+4|(1+(2+\beta-\gamma)t)^m}\right\}. \quad (0.27)$$

Proof. Since $h \in \mathcal{UAR}_{\Sigma}^{\lambda}(\phi)$, there exist two analytic functions $u, v: \Delta \to \Delta$ with $|u(z)| < 1, |v(w)| <$

$1, u(0) = v(0) = 0$ and a function ψ defined by (0.3) satisfies:

$$\left(\frac{z(D^m(\gamma,\beta,t)f(z))'}{D^m(\gamma,\beta,t)f(z)}\right)^{\lambda}\left(1+\frac{z(D^m(\gamma,\beta,t)f(z))''}{(D^m(\gamma,\beta,t)f(z))'}\right)^{1-\lambda} - 1 = [\phi(u(z)) - 1]\psi(z) \qquad (0.28)$$

and

$$\left(\frac{z(D^m(\gamma,\beta,t)g(w))'}{D^m(\gamma,\beta,t)g(w)}\right)^{\lambda}\left(1+\frac{w(D^m(\gamma,\beta,t)g(w))''}{(D^m(\gamma,\beta,t)g(w))'}\right)^{1-\lambda} - 1 = [\phi(v(w)) - 1]\psi(w). \qquad (0.29)$$

Consider functions p and q such that

$$p(z) = \frac{1+u(z)}{1-u(z)} = 1 + \sum_{j=1}^{\infty} d_j z^j$$

equivalently

$$u(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[d_1 z + \left(d_2 - \frac{d_1^2}{2}\right)z^2 + \cdots\right] \qquad (0.30)$$

and

$$q(w) = \frac{1+v(w)}{1-v(w)} = 1 + \sum_{j=1}^{\infty} e_j w^j$$

equivalently

$$v(w) = \frac{q(w)-1}{q(w)+1} = \frac{1}{2}\left[e_1 w + \left(e_2 - \frac{e_1^2}{2}\right)w^2 + \cdots\right]. \qquad (0.31)$$

Clearly p and q are analytic in Δ with $p(0) = q(0) = 1$ and have their positive real part in Δ. Now using equations (0.30) and (0.31), R.H.S. of equations (0.28) and (0.29) can be expressed as

$$[\phi(u(z)) - 1]\psi(z) = \frac{b_0 c_1 d_1}{2} z + \left\{\frac{b_0 c_1 d_2}{2} - \frac{b_0 c_1 d_1^2}{4} + \frac{b_0 c_2 d_1^2}{4} + \frac{b_1 c_1 d_1}{2}\right\} z^2 + \cdots \qquad (0.32)$$

and

$$[\phi(v(w)) - 1]\psi(w) = \frac{b_0 c_1 e_1}{2} w + \left\{\frac{b_0 c_1 e_2}{2} - \frac{b_0 c_1 e_1^2}{4} + \frac{b_0 c_2 e_1^2}{4} + \frac{b_1 c_1 e_1}{2}\right\} w^2 + \cdots. \qquad (0.33)$$

By considering functions h and g given by equations (0.1) and (0.2), L.H.S. of equations (0.28) and (0.29) can be expressed as

$$\left(\frac{z(D^m(\gamma,\beta,t)h(z))'}{D^m(\gamma,\beta,t)h(z)}\right)^{\lambda}\left(1+\frac{z(D^m(\gamma,\beta,t)h(z))''}{(D^m(\gamma,\beta,t)h(z))'}\right)^{1-\lambda} - 1$$
$$= (2-\lambda)(1+(1+\beta-\gamma)t)^m a_2 z + \left[(6-4\lambda)(1+(2+\beta-\gamma)t)^m a_3 + \left(\frac{\lambda^2+5\lambda-8}{2}\right)(1+(1+\beta-\gamma)t)^{2m} a_2^2\right] z$$
(0.34)

and

$$\left(\frac{z(D^m(\gamma,\beta,t)g(w))'}{D^m(\gamma,\beta,t)g(w)}\right)^{\lambda}\left(1+\frac{w(D^m(\gamma,\beta,t)g(w))''}{(D^m(\gamma,\beta,t)g(w))'}\right)^{1-\lambda} - 1$$
$$= -(2-\lambda)(1+(1+\beta-\gamma)t)^m a_2 w + \left[\left(\frac{\lambda^2-11\lambda+16}{2}\right)(1+(1+\beta-\gamma)t)^{2m} a_2^2 + (4\lambda-6)(1+(2+\beta-\gamma)t)^m a\right.$$
(0.35)

Using equations (0.32), (0.33), (0.34) and (0.35) and equating coefficients of like powers of z and w (only first two terms), we get

$$(2-\lambda)(1+(1+\beta-\gamma)t)^m a_2 = \frac{b_0 c_1 d_1}{2}, \qquad (0.36)$$

$$(6-4\lambda)(1+(2+\beta-\gamma)t)^m a_3 + \left(\frac{\lambda^2+5\lambda-8}{2}\right)(1+(1+\beta-\gamma)t)^{2m} a_2^2 = \frac{b_0 c_1 d_2}{2} - \frac{b_0 c_1 d_1^2}{4} + \frac{b_0 c_2 d_1^2}{4} +$$

$\frac{b_1c_1d_1}{2}$, (0.37)

$$-(2-\lambda)(1+(1+\beta-\gamma)t)^m a_2=\frac{b_0c_1e_1}{2} \quad (0.38)$$

and

$$\left(\frac{\lambda^2-11\lambda+16}{2}\right)(1+(1+\beta-\gamma)t)^{2m}a_2^2+(4\lambda-6)(1+(2+\beta-\gamma)t)^m a_3=\frac{b_0c_1e_2}{2}-\frac{b_0c_1e_1^2}{4}+\frac{b_0c_2e_1^2}{4}+\frac{b_1c_1e_1}{2}. \quad (0.39)$$

From equations (0.36) and (0.38), we get

$$d_1=-e_1 \quad (0.40)$$

and

$$8(2-\lambda)^2(1+(1+\beta-\gamma)t)^{2m}a_2^2=b_0^2c_1^2(d_1^2+e_1^2). \quad (0.41)$$

By adding (0.37) and (0.39) in light of (0.40), we get

$$2(\lambda^2-3\lambda+4)(1+(1+\beta-\gamma)t)^{2m}a_2^2=b_0c_1(d_2+e_2)+b_0e_1^2(c_2-c_1). \quad (0.42)$$

By applying lemma (0.2) to equations (0.41) and (0.42), we get the desire result (0.26).

By subtracting (0.39) from (0.37) in light of (0.40), we get

$$a_3=\frac{(1+(1+\beta-\gamma)t)^{2m}}{(1+(2+\beta-\gamma)t)^m}a_2^2+\frac{b_0c_1(d_2-e_2)+2b_1c_1e_1}{8(3-2\lambda)(1+(2+\beta-\gamma)t)^m}. \quad (0.43)$$

Using equations (0.41) and (0.43), we get

$$a_3=\frac{b_0^2c_1^2(d_1^2+e_1^2)}{8(2-\lambda)^2(1+(2+\beta-\gamma)t)^m}+\frac{b_0c_1(d_2-e_2)+2b_1c_1e_1}{8(3-2\lambda)(1+(2+\beta-\gamma)t)^m}. \quad (0.44)$$

Using equations (0.22) and (0.23), we get

$$a_3=\frac{b_0c_1(d_2+e_2)+b_0e_1^2(c_2-c_1)}{2(\lambda^2-3\lambda+4)(1+(2+\beta-\gamma)t)^m}+\frac{b_0c_1(d_2-e_2)+2b_1c_1e_1}{8(3-2\lambda)(1+(2+\beta-\gamma)t)^m}. \quad (0.45)$$

By applying lemma (0.2) to equations (0.44) and (0.45), we get the desire result (0.27). This completes the proof of Theorem (0.11).

4. References

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