Fixed Point Theorems by Generalized distance Function in Ordered Metric Spaces

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Abstract:
Fixed point theory in partially ordered metric spaces has greatly developed in recent times. In this article we prove certain fixed point theorems for multi valued and single valued mappings in such spaces, by using altering distance function. Our results extend some existing results.

Keywords: Partially ordered set, Multi valued mapping, single valued mapping, fixed point.

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Introduction and Preliminaries

Let (X, d) be a metric space. We denote the class of non empty and bounded subsets of X by $B(X)$. For $A, B \in B(X)$, function $D(A, B)$ and $\delta(A, B)$ are defined as follows:

$$D(A,B) = inf\{d(a,b) : a \in A, b \in B\}$$
$$\delta(A,B) = sup\{d(a,b) : a \in A, b \in B\}$$

If $A = \{a\}$ then we write $D(A,B) = D(a,B)$ and $\delta(A,B) = \delta(a,B)$. Also in addition, if $B = \{b\}$, then $D(A,B) = d(a,b)$ and $\delta(A,B) = d(a,b)$. Obviously, $D(A,B) \leq \delta(A,B)$. For all $A, B, C \in B(X)$, the definition of $\delta(A,B)$ yields the following:

$$\delta(A,B) = \delta(B,A)$$
$$\delta(A,B) \leq \delta(A,C) + \delta(C,B)$$

$\delta(A,B) = 0$ iff $A = B = \{a\}$
$\delta(A,B) = diam\, A$ (Fisher 1981, and Iseki, 1983).

Fixed point for multivalued functions is a vast chapter of functional analysis. In particular, the function $\delta(A,B)$ has been used in many works in this area. Some of these works are noted in Choudhury [3], Fisher [7] and Fisher and Ise'ki [8].

We will use the following relation between two non empty subsets of a partially ordered set.

Definition 1: (Beg and Butt, [2]) : Let A and B be two non empty subsets of a partially ordered set $(X, \preccurlyeq)$. The relation between A and B is denoted and defined as follows:

$A \preccurlyeq B$, if for every $a \in A$ there $exists$ $b \in B$ such that $a \preccurlyeq b$.

In 1984, M.S. Khan, M. Swalech and S. Sessa [9] expanded the research of the metric fixed point theory to a new category by introducing a control function which they called an altering distance function.

We will utilize the following control function which is also referred to a Altering distance function.

Definition 2 : (Khan et al. [9]): A function $\psi : [0, \infty) \to [0, \infty)$ is called a Altering distance function if the following properties are satisfied:

i. ψ is monotone increasing and continuous,
ii. $\psi(t) = 0$ if and only if $t = 0$.

The above control function has been utilized in a large number of works in metric fixed point theory. Some recent references are Choudhury[4], Doric [5], Dutta and Choudhury [6], Naidu [10] and Sastry and Babu [11]. This control function has also been extended and applied to fixed point problems in probabilistic metric spaces, and fuzzy metric spaces.

The purpose of this paper is to establish the existence of fixed point if multivalued mappings in partially ordered metric spaces. The mappings are assumed to satisfy certain inequalities which involved the above mentioned control functions. Further we have established that in the corresponding singlevalued cases of partial ordered condition of the metric space can be omitted if the function is continuous.

Main Results

Theorem 2.1: Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to B(X)$ be a multivalued mapping such that the following conditions are satisfied;

1. there exists $x_0 \in X$ such that $\{x_0\} \preccurlyeq Tx\_0$,
2. for $x, y \in X, x \preccurlyeq y$ implies $Tx \preccurlyeq Ty$,
3. if $x_n \to x$ is a non decreasing sequence in X, then $x_n \preccurlyeq x$ for all n,
4. $\psi\big(\delta(Tx, Ty)\big) \le \alpha\, \psi(max\,\{D(x, Tx), D(y, Ty)\})$
$$+\beta\, \psi\,(max\,\{D(x, Ty), D(y, Tx)\}) + \gamma\, \psi\,(d(x, y))$$

For all comparable $x, y \in X$ where $\alpha, \beta, \gamma \in (0,1)$ such that $0 < \alpha + 2\beta + \gamma < 1$ and ψ is an altering distance function. Then T has a fixed point.

Proof: By the assumption (i) there exists $x_1 \in Tx_0$ such that $x_0 \preccurlyeq x_1$. By the assumption (ii), $Tx_0 \preccurlyeq Tx_1$. Then there $exists\ x_2 \in Tx_1$ such that $x_1 \preccurlyeq x_2$. Continuing the process we construct a monotone increasing sequence $\{x_n\}$ in X such $that\ x_{n+1} \in Tx_n$ for all $n \ge 0$. Thus we have $x_0 \preccurlyeq x_1 \preccurlyeq x_2 \preccurlyeq x_3 \preccurlyeq \ldots\ldots \preccurlyeq x_n \preccurlyeq x_{n+1} \preccurlyeq \ldots.$.

If there exists a positive integer N such that $x_N = x_{N+1}$, then x_N is a fixed point of T. Hence we shall assume that $x_n \neq x_{n+1}$ for all $n \ge 0$.

Using the monotone property $of\ \psi$ and the condition (iv), we have for all $n \ge 0$,

$$\psi\left(d(x_{n+1}, x_{n+2})\right) \le \psi\big(\delta\,(Tx_n, Tx_{n+1})\big)$$

$$\psi\left(\delta\,(Tx_n, Tx_{n+1})\right) \le \alpha\, \psi(max\,\{D(x_n, Tx_n), D(x_{n+1}, Tx_{n+1})\})$$

$$+\beta\, \psi(max\,\{D(x_n, Tx_{n+1}), D(x_{n+1}, Tx_n)\}) + \gamma\psi\left(d(x_n, x_{n+1})\right)$$

$$\psi\left(d(x_{n+1}, x_{n+2})\right) \le \alpha\, \psi(max\,\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\})$$

$$+\beta\, \psi\,(max\,\{d(x_n, x_{n+2}), d(x_{n+1}, x_{n+1})\}) + \gamma\, \psi\left(d(x_n, x_{n+1})\right)$$

There arise two cases.

Case - 1, if we take $max\,\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = d(x_n, x_{n+1})$ then,

$$\psi\left(d(x_{n+1},x_{n+2})\right) \leq \frac{\alpha+\beta+\gamma}{1-\beta}\ \psi\left(d(x_n,x_{n+1})\right)$$

Case - 2, if we take $max\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\} = d(x_{n+1},x_{n+2})$ then,

$$\psi\left(d(x_{n+1},x_{n+2})\right) \leq \frac{\beta+\gamma}{1-\alpha-\beta}\ \psi\left(d(x_n,x_{n+1})\right)$$

Since $0 < \alpha + 2\beta + \gamma < 1$ in both cases, which implies

$\psi\left(d(x_{n+1},x_{n+2})\right) \leq k\ \psi\left(d(x_n,x_{n+1})\right)$

where $k = max\left\{\frac{\beta+\gamma}{1-\alpha-\beta},\frac{\alpha+\beta+\gamma}{1-\beta}\right\}$.

Therefore, $d(x_{n+1},x_{n+2}) < d(x_n,x_{n+1})$ for all $n \geq 0$ and $\{d(x_n,x_{n+1})\}$ is monotone decreasing sequence of non negative real numbers. Hence there exists an $r \geq 0$ such that,

$d(x_n,x_{n+1}) \to r\ \ as\ \ n \to \infty$.

Taking the limit as $n \to \infty$ in (2.1) and using the continuity of ψ, we have

$$\psi(r) \leq k\,\psi(r)$$

which is a contradiction unless $r = 0$.

Hence,

$lim_{n\to\infty}\ d(x_n,x_{n+1}) = 0$

Next we show that $\{x_n\}$ is a Cauchy sequence. If otherwise, there exists an $\epsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}\ and\ \{n(k)\}$ such that for all positive integers k, $n(k) > m(k) > k$ and $\left(x_{m(k)},x_{n(k)}\right) \geq \epsilon$.

Assume that n(k) is the smallest such positive integer, we get, $n(k) > m(k) > k$

$d\left(x_{m(k)},x_{n(k)}\right) \geq \epsilon$ and $d\left(x_{m(k)},x_{n(k)-1}\right) < \epsilon$.

Now,

$$\epsilon \leq d\left(x_{m(k)},x_{n(k)}\right) \leq d\left(x_{m(k)},x_{n(k)-1}\right) + d(x\_(n(k)-1),x_{n(k)})$$

that is,

$$\epsilon \leq d\left(x_{m(k)},x_{n(k)}\right) \leq \epsilon + d\left(x_{n(k)-1},x_{n(k)}\right)$$

Taking the limit as $k \to \infty$ in the above inequality and (2.3), we have

$lim_{n\to\infty}\ d\left(x_{m(k)},x_{n(k)}\right) = \epsilon$

Again,

$$d\left(x_{m(k)},x_{n(k)}\right) \leq d\left(x_{m(k)},x_{m(k)+1}\right) + d\left(x_{m(k)+1},x_{n(k)+1}\right) + d\left(x_{n(k)+1},x_{n(k)}\right)$$

and,

$$d\left(x_{m(k)+1},x_{n(k)+1}\right) \leq d\left(x_{m(k)+1},x_{m(k)}\right) + d\left(x_{m(k)},x_{n(k)}\right) + d\left(x_{n(k)},x_{n(k)+1}\right)$$

Taking the limit as $k \to \infty$ in the above inequality and (2.3) and (2.4), we have,

$lim_{n\to\infty}\ d\left(x_{m(k)+1},x_{n(k)+1}\right) = \epsilon$

Again,

$$d\left(x_{m(k)},x_{n(k)}\right) \leq d\left(x_{m(k)},x_{n(k)+1}\right) + d\left(x_{n(k)+1},x_{n(k)}\right)$$

and,

$$d\left(x_{m(k)},x_{n(k)+1}\right) \leq d\left(x_{m(k)},x_{n(k)}\right) + d\left(x_{n(k)},x_{n(k)+1}\right)$$

Taking the limit $as\ k \to \infty$ in the above inequality and (2.3) and (2.4), we have,

$lim_{n\to\infty}\ d\big(x_{m(k)}, x_{n(k)+1}\big)\ = \epsilon$

Similarly we have that
$lim_{n\to\infty}\ d\big(x_{n(k)}, x_{m(k)+1}\big)\ = \epsilon$
For each positive integer k, $x_{m(k)}$ and $x_{n(k)}$ are comparable. Then using the monotone property of ψ and the condition (iv), we have

$$\psi\ \Big(d\big(x_{m(k)+1}, x_{n(k)+1}\big)\Big) \le\ \psi\Big(\delta\ \big(Tx_{m(k)}, Tx_{n(k)}\big)\Big)$$

$$\psi\Big(\delta\ \big(Tx_{m(k)}, Tx_{n(k)}\big)\Big) \le \alpha\ \psi\big(max\ \{D\big(x_{m(k)}, Tx_{m(k)}\big), D\big(x_{n(k)}, Tx_{n(k)}\big)\}\big)$$

$$+\beta\ \psi\ \big(max\ \{\ D\big(x_{m(k)}, Tx_{n(k)}\big), D\big(x_{n(k)}, Tx_{m(k)}\big)\}\big)\ + \gamma\ \psi\ \Big(d\big(x_{m(k)}, x_{n(k)}\big)\Big)$$

By using (iv) and on taking limit as $k \to \infty$ in the above inequality and (2.3) - (2.7), and using the continuity of ψ we have,

$$\psi\ (\epsilon) \le\ k\ \psi\ (\epsilon)$$

which is contradiction by virtue of a property of ψ.
$Hence$ $\{\ x_n\}$ is a Cauchy sequence. From the completeness of X, there exists a $z \in\ X$ such that
$x_n \to\ z\ \ as\ \ n \to \infty$
By the assumption (iii), $x_n \preccurlyeq\ z$, for all n.
Then by the monotone property of ψ and the condition (iv), we have

$$\psi\ \big(d(x_{n+1}, Tz)\big) \le\ \ \psi\ \Big(\delta\ \big(Tx_n, T(z)\big)\Big)$$

By using (iv) and on taking limit as $k \to \infty$ in the above inequality from (2.3) and (2.8), and using the continuity of ψ we have,

$$\psi\big(\delta\ (z, Tz)\big) \le\ k\ \psi\big((Dz, Tz)\big) \le\ k\ \psi\ \big(\delta\ (z, Tz)\big),$$

which implies that, $\delta\ (z, Tz)\ \ =\ \ 0$ or that $\{\ z\ \} =\ Tz$. Moreover, z is a fixed point of T.

Corollary 2.2: Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to B(X)$ be a multivalued mapping such that the following conditions are satisfied;

1. there exists $x_0 \in X$ such that $\{\ x_0\} \preccurlyeq\ Tx_0$,
2. for $x, y \in\ X, x \preccurlyeq\ y$ implies $Tx \preccurlyeq Ty$,
3. if $x_n \to\ x$ is a non decreasing sequence in X, then $x_n \preccurlyeq\ x$ for all n,
4. $\delta(Tx, Ty) \le \alpha$ max$\{\ D(x, Tx), D(y, Ty)\}$

$$+\beta\ max\ \{\ D(x, Ty), D(y, Tx)\}\ + \gamma\ d(x, y)$$

For all comparable $x, y \in\ X$ where $\alpha\ , \beta\ , \gamma \in\ (0,1)$ such that $0\ < \alpha\ +\ 2\beta + \gamma\ <\ 1$ and $\backslash psi$ is an altering distance function. Then T has a fixed point.

Proof: On takeing an identity function in Theorem 2.1, then the above result is true and noting to prove.
The following corollary is a spacial case of Theorem 2.1 when T is a singlevalued mapping.

Corollary 2.3: Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T :\ X \to\ X$ be a mapping such that the following conditions are satisfied;

1. there exists $x_0 \in\ X$ such that $\{\ x_0\} \preccurlyeq Tx_0$,
2. for $x, y \in\ X, x \preccurlyeq\ y\ implies\ Tx \preccurlyeq\ Ty$,
3. if $x_n \to\ x$ is a non decreasing sequence in X, then $x_n \preccurlyeq\ x$ for all n,

4. $\psi\big(d(Tx,Ty)\big) \le \alpha\, \psi(\max\{ d(x,Tx), d(y,Ty)\})$
$$+\beta\, \psi\,(max\,\{d(x,Ty), d(y,Tx)\}\,) + \gamma\, \psi\,\big(d(x,y)\big)$$

For all comparable $x, y \in X$ where $\alpha, \beta, \gamma \in (0,1)$ such that $0 < \alpha + 2\beta + \gamma < 1$ and ψ is an altering distance function. Then T has a fixed point.

In the following theorem we replace condition (iii) of the above corollary by requiring T to be continuous.

Theorem 2.4: Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a mapping such that the following conditions are satisfied;

1. there exists $x_0 \in X$ such that $\{x_0\} \preccurlyeq Tx_0$,
2. for $x, y \in X, x \preccurlyeq y$ implies $Tx \preccurlyeq Ty$,
3. $\psi\big(d(Tx,Ty)\big) \le \alpha\, \psi(max\,\{ d(x,Tx), d(y,Ty)\})$
$$+\beta\, \psi\,(max\,\{ d(x,Ty), d(y,Tx)\}) + \gamma\, \psi\,(d(x,y))$$

For all comparable $x, y \in X$ where $\alpha, \beta, \gamma \in (0,1)$ such that $0 < \alpha + 2\beta + \gamma < 1$ and ψ is an altering distance function. Then T has a fixed point.

Proof: We can treat T as a multivalued mapping in which case Tx is a singleton set for every $x \in X$. Then we consider the same sequence $\{x_n\}$ as in the proof of Theorem 2.1, Arguing exactly as in the proof of Theorem 2.1, we have that $\{x_n\}$ is a Cauchy sequence and $lim_{n\to\infty}\,(x_n) = z$. Then the continuity of T implies that,

$$z = lim_{n\to\infty}\,(x_{n+1}) = lim_{n\to\infty}\, T(x_n) = Tz$$

and this proves that z is a fixed point of T.

Theorem 2.5: Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to B(X)$ be a multivalued mapping such that the following conditions are satisfied;

1. there exists $x_0 \in X$ such that $\{x_0\} \preccurlyeq$ Tx_0 ,
2. for $x, y \in X, x \preccurlyeq y$ implies $Tx \preccurlyeq Ty$,
3. if $x_n \to x$ is a non decreasing sequence in X, then $x_n \preccurlyeq x$ for all n,
4. $\psi\big(\delta(Tx,Ty)\big) \le \psi(max\,\{ D(x,Tx), D(y,Ty)\})$

$$+\ \psi\,(max\,\{ D(x,Ty), D(y,Tx)\}) + \psi\,\big(d(x,y)\big)$$
$$-\phi(\,max\,\{\,\delta(x,Tx), \delta(y,Ty), \delta(x,Ty), \delta(y,Tx), d(x,y)\})$$

For all comparable $x, y \in X$ where ψ is an altering distance function $and\ \phi : [0,\infty) \to [0,\infty)$ is any continuous function $with\ \phi\,(t) = 0$ if and only $if\ t = 0$. Then T has a fixed point.

Proof: We take the same sequence $\{x_n\}$ as in the proof of Theorem 2.1. If there exists a positive integer N such that $x_N = x_{N+1}$, then x_N is a fixed point of T. Hence we shall assume that $x_n \ne x_{n+1}$ for all $n \ge 0$.

Using the monotone property of ψ and the condition (iv), we have for all $n \ge 0$,

$$\psi\big(d(x_{n+1}, x_{n+2})\big) \le \psi\,\big(\delta\,(Tx_n, Tx_{n+1})\big)$$

$$\psi\big(\delta(Tx_n, Tx_{n+1})\big)$$
$$\le \psi(max\,\{ D(x_n, Tx_n), D(x_{n+1}, Tx_{n+1})\})$$
$$+\ \psi\,(max\,\{ D(x_n, Tx_{n+1}), D(x_{n+1}, Tx_n)\})$$
$$+\psi\,\big(d(x_n, x_{n+1})\big) - \phi(\,max\,\{\delta(x_n, Tx_n), \delta(\,x_{n+1}, Tx_{n+1}), \delta(x_n, Tx_{n+1}), \delta(x_{n+1}, Tx_n), d(x_n, x_{n+1})\,\})$$

$$\psi\left(d(x_{n+1},x_{n+2})\right)$$
$$\leq \psi(max\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\}) + \psi(max\{d(x_n,x_{n+2}),d(x_{n+1},x_{n+1})\}) + \psi\left(d(x_n,x_{n+1})\right)$$
$$-\phi(max\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2}),d(x_n,x_{n+2}),d(x_{n+1},x_{n+1}),d(x_n,x_{n+1})\})$$
$$\psi d(x_{n+1},x_{n+2}) \leq \psi(max\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\}) + \psi\left(d(x_n,x_{n+2})\right) + \psi\left(d(x_n,x_{n+1})\right)$$
$$-\phi(max\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2}),d(x_n,x_{n+2}),\})$$

Then from the above inequality we have,

$$\psi\left(d(x_{n+1},x_{n+2})\right) \leq \psi\left(d(x_{n+1},x_{n+2})\right) - \phi\left(d(x_{n+1},x_{n+2})\right)$$

that is , $\phi\left(d(x_{n+1},x_{n+2})\right) \leq 0$ which implies that $\left(d(x_{n+1},x_{n+2})\right) = 0$, or that $x_{n+1} = x_{n+2}$, contradicting our assumption that is $x_n \neq x_{n+1}$ for each n.

Therefore, $d(x_{n+1},x_{n+2}) < d(x_n,x_{n+1})$ for all $n \geq 0$ and $\{d(x_n,x_{n+1})\}$ is monotone decreasing sequence of non negative real numbers. Hence there exists $an\ r \geq 0$ such that,

$d(x_n,x_{n+1}) \to r \quad as \quad n \to \infty$

Taking the limit $as\ n \to \infty$ and using the continuity of ψ, we have

$$\psi(r) \leq \psi(r) - \phi(r)$$

which is a contradiction $unless\ r = 0$.

Hence,

$lim_{n\to\infty} d(x_n,x_{n+1}) = 0$

Next we show that $\{x_n\}$ is a Cauchy sequence. If not then using an argument to that given in Theorem 2.1, we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ for which,

$lim_{k\to\infty} d\left(x_{m(k)},x_{n(k)}\right) = \epsilon$

$lim_{k\to\infty} d\left(x_{m(k)+1},x_{n(k)+1}\right) = \epsilon$

$lim_{k\to\infty} d\left(x_{m(k)},x_{n(k)+1}\right) = \epsilon$

$lim_{k\to\infty} d\left(x_{n(k)},x_{m(k)+1}\right) = \epsilon$

for each positive integer k, $x_{m(k)}, x_{n(k)}$ are comparable. Then using monotone property of ψ and the condition (iv), we have

$$\psi\left(d\left(x_{m(k)+1},x_{n(k)+1}\right)\right) \leq \psi\left(\delta\left(Tx_{m(k)},Tx_{n(k)}\right)\right)$$

$$\psi\left(\delta\left(Tx_{m(k)},Tx_{n(k)}\right)\right) \leq \psi(max\{D\left(x_{m(k)},Tx_{m(k)}\right),D\left(x_{n(k)},Tx_{n(k)}\right)\})$$
$$+ \psi(max\{D\left(x_{m(k)},Tx_{n(k)}\right),D\left(x_{n(k)},Tx_{m(k)}\right)\})$$
$$+\psi\left(d\left(x_{m(k)},x_{n(k)}\right)\right)$$
$$-\phi\left(\max\left\{\begin{matrix}\delta\left(x_{m(k)},Tx_{m(k)}\right),\delta\left(x_{n(k)},Tx_{n(k)}\right),\delta\left(x_{m(k)},Tx_{n(k)}\right),\\ \delta\left(x_{n(k)},Tx_{m(k)}\right),d\left(x_{m(k)},x_{n(k)}\right)\end{matrix}\right\}\right)$$

$$\psi\left(d\left(x_{m(k)+1},x_{n(k)+1}\right)\right) \leq \psi(max\{d\left(x_{m(k)},x_{m(k)+1}\right),d\left(x_{n(k)},x_{n(k)+1}\right)\})$$
$$+ \psi(max\{d\left(x_{m(k)},x_{n(k)+1}\right),d\left(x_{n(k)},x_{m(k)+1}\right)\})$$
$$+\psi\left(d\left(x_{m(k)},x_{n(k)}\right)\right)$$
$$-\phi\left(max\left\{\begin{matrix}d\left(x_{m(k)},x_{m(k)+1}\right),d\left(x_{n(k)},x_{n(k)+1}\right),d\left(x_{m(k)},x_{n(k)+1}\right),\\ d\left(x_{n(k)},x_{m(k)+1}\right),d\left(x_{m(k)},x_{n(k)}\right)\end{matrix}\right\}\right)$$

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Letting $k \to \infty$ in the above inequality, using (2.10)-(2.14) and the continuous of ψ and ϕ, we have

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon)$$

which contradiction by virtue of the property of ϕ .
Hence $\{x_n\}$ is Cauchy sequence. From the completeness of X, there exists a $z \in X$ such that,
$x_n \to z \; as \quad n \to \infty$
by the assumption of (iii), $x_n \preccurlyeq z$, for all n,
Then by the monotone property of ψ and the condition (iv), we have

$$\psi\left(d(x_{n+1}, Tz)\right) \leq \psi\left(\delta\left(Tx_n, T(z)\right)\right)$$

$$\psi\left(\delta\left(Tx_n, T(z)\right)\right)$$
$$\leq \psi(max\{D(x_n, Tx_n), D((z), T(z))\}) + \psi(max\{D(x_n, T(z)), D((z), Tx_n)\})$$
$$+\psi\left(d(x_n, (z))\right) - \phi\left(max\left\{\begin{matrix}\delta(x_n, Tx_n), \delta((z), T(z)), \delta(x_n, T(z)), \\ \delta((z), Tx_n), d(x_n, (z))\end{matrix}\right\}\right)$$

Taking the limit as $n \to \infty$ in the above inequality and (2.10) and (2.15), we have,

$$\psi\left(\delta\left(z, T(z)\right)\right) \leq \psi(D(z, Tz)) - \phi\left(\delta\left(z, T(z)\right)\right)$$

which implies that,

$$\psi\left(\delta(z, T(z))\right) \leq \psi(\delta(z, T(z))) - \phi\left(\delta(z, T(z))\right)$$

Which is contradiction unless $\delta(z, T(z)) = 0$ or that, $z = Tz$; that is Z is a fixed point of T.

On taking ψ an identity function in Theorem 2.5, we have the following result.
Corollary 2.6: Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to B(X)$ be a multivalued mapping such that the following conditions are satisfied;

1. there exists $x_0 \in X$ such that $\{x_0\} \preccurlyeq Tx_0$,
2. for $x, y \in X, x \preccurlyeq y$ implies $Tx \preccurlyeq Ty$,
3. if $x_n \to x$ is a non decreasing sequence in X, then $x\_n \backslash preceq\; x$ for all n,
4. $\delta(Tx, Ty) \leq \max\{D(x, Tx), D(y, Ty)\}$

$+ \max\{D(x, Ty), D(y, Tx)\} + \left(d(x, y)\right) - \phi(max\{\delta(x, Tx), \delta(y, Ty), \delta(x, Ty), \delta(y, Tx), d(x, y)\})$
For all comparable $x, y \in X$ where ψ is an altering distance function and $\phi : [0, \infty) \to [0, \infty)$ is any continuous function with $\phi(t) = 0$ if and only if $t = 0$. Then T has a fixed point.

The following corollary is a special case of Theorem 2.5 when T is a singlevalued mapping.

Corollary 2.7: Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a multivalued mapping such that the following conditions are satisfied;

1. there exists $x_0 \in X$ such that $\{x_0\} \preccurlyeq Tx_0$,
2. for $x, y \in X, x \preccurlyeq y$ implies $Tx \preccurlyeq Ty$,
3. $if\; x_n \to x$ is a non decreasing sequence in X, then $x_n \preccurlyeq x$ for all n,
4. $\psi\left(d(Tx, Ty)\right) \leq \psi(max\{d(x, Tx), d(y, Ty)\}) + \psi(max\{d(x, Ty), d(y, Tx)\})$
$+\psi(d(x, y)) - \phi(max\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), d(x, y)\})$

For all comparable $x, y \in X$ where ψ is an altering distance function and $\phi : [0, \infty) \to [0, \infty)$ is any continuous function with $\phi(t) = 0$ if and only if $t = 0$. Then T has a fixed point.

In the following theorem we replace condition (iii) of the above corollary by requiring T to be continuous.

Theorem 2.8: Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a multivalued mapping such that the following conditions are satisfied;

1. there exists $x_0 \in X$ such that $\{x_0\} \preccurlyeq Tx\_0$,
2. for $x, y \in X, x \preccurlyeq y$ implies $Tx \preccurlyeq Ty$,
3. $\psi(d(Tx, Ty)) \leq \psi(max\{d(x, Tx), d(y, Ty)\}) + \psi(max\{d(x, Ty), d(y, Tx)\})$
   $+\psi(d(x, y)) - \phi(max\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), d(x, y)\})$

For all comparable $x, y \in X$ where ψ is an altering distance function and $\phi : [0, \infty) \to [0, \infty)$ is any continuous function with $\phi(t) = 0$ if and only if $t = 0$. Then T has a fixed point.

Proof: We can treat T as a multivalued mapping in which $case\ Tx$ is a singleton set for every $x \in X$. Then we consider the same sequence $\{x_n\}$ as in the proof of Theorem 2.5, Arguing exactly as in the proof of Theorem 2.5, we have that $\{x_n\}$ is a Cauchy sequence and $lim_{n\to\infty}(x_n) = z$. Then the continuity of T implies that,

$$z = lim_{n\to\infty}(x_{n+1}) = lim_{n\to\infty} T(x_n) = Tz$$

and this proves that z is a fixed point of T.

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